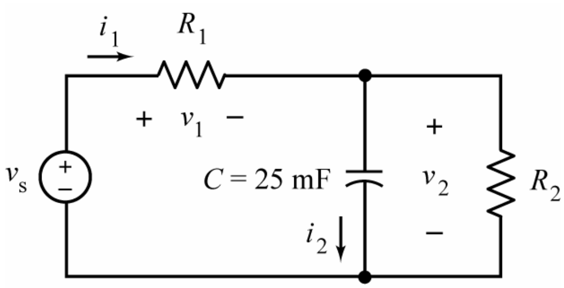
Kirchhoff's Laws in Dynamic Circuits

Dynamic circuits are circuits that contain capacitors and inductors. Later we will learn to analyze some dynamic circuits by writing and solving differential equations. In these notes, we consider some simpler examples that can be solved using only Kirchhoff's laws and the element equations of the capacitor and the inductor.

Example 1:
Consider this circuit



Additionally, we are given the following representations of the voltage source voltage and one of the resistor voltages:

$$v_s = \begin{cases} 10 \text{ V} & \text{for} \quad t < 0 \\ 20 \text{ V} & \text{for} \quad t > 0 \end{cases} \qquad \text{and} \qquad v_1 = \begin{cases} 2 \text{ V} & \text{for} \quad t < 0 \\ 8e^{-5t} + 4 \text{ V} & \text{for} \quad t > 0 \end{cases}$$

We wish to express the capacitor current, i_2, as a function of time, t.

Plan:
First, apply Kirchhoff's voltage law (KVL) to the loop consisting of the source, resistor R_1 and the capacitor to determine the capacitor voltage, v_2, as a function of time, t. Next, use the element equation of the capacitor to determine the capacitor current as a function of time, t.

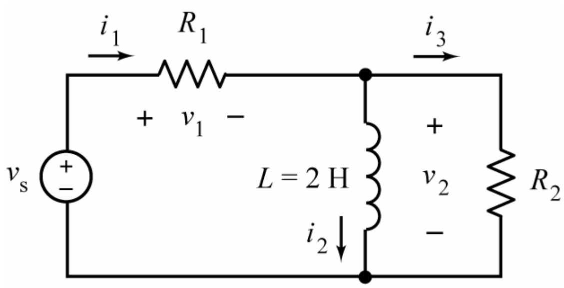
Solution:
Apply Kirchhoff's voltage law (KVL) to the loop consisting of the source, resistor R_1 and the capacitor to write

$$v_1 + v_2 - v_s = 0 \quad \Rightarrow \quad v_2 = v_s - v_1 = \begin{cases} 8 \text{ V} & \text{for} \quad t < 0 \\ 16 - 8e^{-5t} \text{ V} & \text{for} \quad t > 0 \end{cases}$$

Use the element equation of the capacitor to write

$$i_2 = C\frac{d\,v_2}{d\,t} = 0.025\frac{d\,v_2}{d\,t} = \begin{cases} 0 \text{ A} & \text{for} \quad t < 0 \\ 0.025\dfrac{d}{d\,t}\left(16 - 8e^{-5t}\right) & \text{for} \quad t > 0 \end{cases} = \begin{cases} 0 \text{ A} & \text{for} \quad t < 0 \\ 1e^{-5t} \text{ A} & \text{for} \quad t > 0 \end{cases}$$

Example 2:
Consider this circuit



where the resistor currents are given by

$$i_1 = \begin{cases} 0.8 \text{ A} & \text{for} \quad t < 0 \\ 0.8e^{-2t} - 0.8 \text{ A} & \text{for} \quad t > 0 \end{cases} \qquad \text{and} \qquad i_3 = \begin{cases} 0 \text{ A} & \text{for} \quad t < 0 \\ -0.8e^{-2t} \text{ A} & \text{for} \quad t > 0 \end{cases}$$

Express the inductor voltage, v_2, as a function of time, *t*.

Plan:
First, apply Kirchhoff's current law (KCL) at the top node of the inductor to determine the inductor current, i_2, as a function of time, *t*. Next, use the element equation of the inductor to determine the inductor voltage as a function of time, *t*.

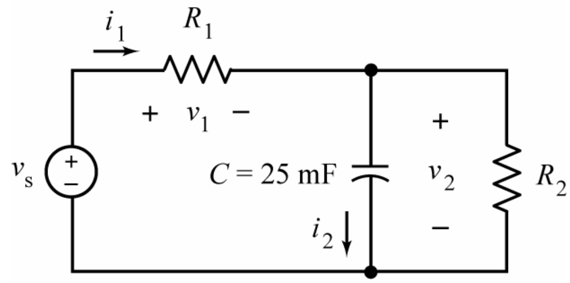
Solution:
Apply Kirchhoff's current law (KCL) at the top node of the inductor to write

$$i_1 = i_2 + i_3 = 0 \quad \Rightarrow \quad i_2 = i_1 - i_3 = \begin{cases} 0.8 \text{ V} & \text{for} \quad t < 0 \\ 1.6e^{-2t} - 0.8 \text{ A} & \text{for} \quad t > 0 \end{cases}$$

Use the element equation of the inductor to write

$$v_2 = L\frac{d\,i_2}{d\,t} = 2\frac{d\,i_2}{d\,t} = \begin{cases} 0 \text{ V} & \text{for} \quad t < 0 \\ 2\dfrac{d}{d\,t}\left(1.6e^{-2t} - 0.8\right) & \text{for} \quad t > 0 \end{cases} = \begin{cases} 0 \text{ V} & \text{for} \quad t < 0 \\ -6.4e^{-2t} \text{ V} & \text{for} \quad t > 0 \end{cases}$$

Example 3:
Consider this circuit



where

$$v_s = 12\cos(5t) \text{ V} \qquad \text{and} \qquad v_1 = 8.653\cos(5t + 33.7°) \text{ V}$$

Express the capacitor current, i_2, as a function of time, *t*.

Plan:
This example is very similar to Example 1. In this case, the source voltage is a sinusoidal function of time. Apparently, that causes v_1 to also be a sinusoidal function of time. We will see that v_2 and i_2 are both sinusoidal functions of time. Further, all of these sinusoidal functions have the same frequency as the source voltage.

As in Example 1, we first apply Kirchhoff's voltage law (KVL) to the loop consisting of the source, resistor R_1 and capacitor to determine the capacitor voltage, v_2, as a function of time, *t*. Next, use the element equation of the capacitor to determine the capacitor current as a function of time, *t*.

Solution:
Apply KVL to the loop consisting of the source, resistor R_1 and capacitor to write

$$v_1 + v_2 - v_s = 0 \quad \Rightarrow \quad v_2 = v_s - v_1 = 12\cos(5t) - 8.653\cos(5t + 33.7°)$$

This subtraction requires a couple of trig identities and some effort:

$$\begin{aligned} v_2 = 12\cos(5t) - 8.653\cos(5t + 33.7°) &= 12\cos(5t) - 8.653\left(\cos(33.7°)\cos(5t) + \sin(33.7°)\sin(5t)\right) \\ &= 12\cos(5t) - \left(7.199\cos(5t) + 4.801\sin(5t)\right) \\ &= 4.801\cos(5t) - 4.801\sin(5t) \\ &= \sqrt{4.801^2 + 4.801^2}\cos\left(5t + \tan^{-1}\left(\frac{-4.801}{4.801}\right)\right) \\ &= 6.790\cos(5t - 45°) \text{ V} \end{aligned}$$

Next, use the element equation of the capacitor to write

$$i_2 = C\frac{d\,v_2}{d\,t} = 0.025\frac{d}{d\,t}6.790\cos\left(5t - 45°\right) = 0.025\left(6.790\right)\left(-5\sin\left(5t - 45°\right)\right)$$
$$= 0.849\left(-\sin\left(5t - 45°\right)\right)$$
$$= 0.849\left(-\cos\left(5t - 45° - 90°\right)\right)$$
$$= 0.849\cos\left(5t + 45°\right) \text{ A}$$

The circuit in Example 3 is an example of an “ac circuit”. Let’s talk about ac circuits.

AC Circuits

AC circuits are electric circuits in which

- The voltages of all independent voltage sources are sinusoidal functions of time.
- The currents of all independent current sources are sinusoidal functions of time.
- These voltage source voltages and current source currents all have the same frequency.

The voltages of the independent voltage sources and currents of the independent current sources are sometimes called the inputs to the circuit. Using this terminology we can say that:

> **AC circuits** are electric circuits in which all of the inputs are sinusoids having the same frequency.

Here’s an important property of AC circuits:

> All the element voltages and currents of an AC circuit are sinusoids having the same frequency as the inputs.

The electric power system in the United States can be modeled as an AC circuit having the frequency 60 Hz or 377 rad/s. Consequently, all of the voltages and currents in this circuit have the form $A\cos\left(377t + \theta\right)$. A particular current or voltage can be specified by providing the values of the amplitude, *A*, and the phase angle, θ.

The circuit in Example 3 is an AC circuit having the frequency 5 rad/s.

Sinusoids and Complex Numbers

It’s useful to associate a complex number $A\angle\theta$ with the sinusoid $A\cos\left(377t + \theta\right)$. The complex number $A\angle\theta$ is given in polar form so *A* represents the amplitude of the complex number and θ represents the phase angle of the complex number.

Now, the addition of sinusoids corresponds to the addition of complex numbers:

$$A\cos(377t+\theta)+B\cos(377t+\phi)=C\cos(377t+\gamma) \quad \Leftrightarrow \quad A\angle\theta+B\angle\phi=C\angle\gamma$$

Suppose we need to find the sum of sinusoid, $A\cos(377t+\theta)+B\cos(377t+\phi)$. Instead, we can choose to find the sum of complex numbers, $A\angle\theta+B\angle\phi$. The sum of the complex numbers, $C\angle\gamma$, provides the sum of the sinusoids, $C\cos(377t+\gamma)$.

It's customary to convert complex numbers from polar form to rectangular form before adding them. That is

$$A\angle\theta = a + jb$$

where $A\angle\theta$ is the complex number in polar form and $a+jb$ is the complex number in rectangular form. The conversion from polar form to rectangular form and visa versa is described by

$$a = \text{the real part of } A\angle\theta = A\cos(\theta)$$

$$b = \text{the imaginary part of } A\angle\theta = A\sin(\theta)$$

$$A = \text{the magnitude of } a+jb = \sqrt{a^2+b^2}$$

$$\theta = \text{the angle of } a+jb = \begin{cases} \tan^{-1}\left(\dfrac{b}{a}\right) & \text{when} \quad a>0 \\ 180° - \tan^{-1}\left(\dfrac{b}{-a}\right) & \text{when} \quad a<0 \end{cases}$$

Two complex numbers in rectangular form, $a+jb$ and $c+jd$, are added as

$$(a+jb)+(c+jd)=(a+c)+j(b+d)=e+jf$$

(The real part of $e+jf$ is equal to the sum of the real parts of $a+jb$ and $c+jd$. Also, the imaginary part of $e+jf$ is equal to the sum of the imaginary parts of $a+jb$ and $c+jd$.)

To summarize, to add two complex numbers in polar form, $A\angle\theta$ and $B\angle\phi$, we first convert both complex numbers to rectangular form:

$$A\angle\theta+B\angle\phi=(A\cos\theta+jA\sin\theta)+(B\cos\phi+jB\sin\phi)=(a+jb)+(c+jd)$$

Next, we add the rectangular forms of the complex numbers:

$$(a+jb)+(c+jd)=(a+c)+j(b+d)=e+jf$$

Finally, we convert the sum from rectangular form to polar form:

$$e + j\,f = C\angle\gamma = \begin{cases} \sqrt{e^2+f^2}\angle\tan^{-1}\left(\dfrac{f}{e}\right) & \text{when} \quad e>0 \\ \sqrt{e^2+f^2}\angle\left(180° - \tan^{-1}\left(\dfrac{f}{e}\right)\right) & \text{when} \quad e<0 \end{cases}$$

In Example 3 we needed to calculate the sum of two sinusoids having at the frequency 5 rad/s:

$$12\cos(5t) - 8.653\cos(5t + 33.7°)$$

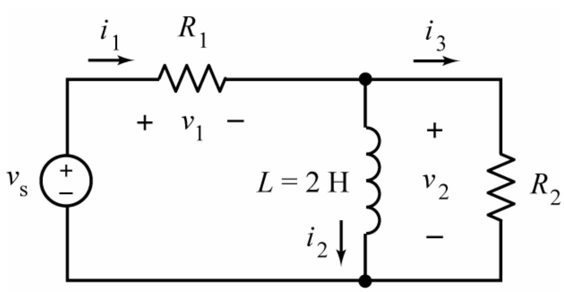
The corresponding sum of complex numbers is

$$\begin{aligned} 12\angle 0 - 8.653\angle 33.7° &= \left(12\cos(0) + j\,12\sin(0)\right) - \left(8.653\cos(33.7°) + j\,8.653\sin(33.7°)\right) \\ &= (12 + j\,0) - (7.199 + j\,4.801) \\ &= (12 - 7.199) + j\,(0 - 4.801) \\ &= 4.801 - j\,4.801 \\ &= \sqrt{4.801^2 + (-4.801)^2}\angle\tan^{-1}\left(\frac{-4.801}{4.801}\right) \\ &= 6.790\angle -45° \end{aligned}$$

The sinusoid at the frequency 5 rad/s corresponding to $6.790\angle -45°$ is $6.790\cos(5t - 45°)$. Consequently

$$12\cos(5t) - 8.653\cos(5t + 33.7°) = 6.790\cos(5t - 45°)$$

Example 4:
Consider this circuit



where the resistor currents are given by

$$i_1 = 1.040\cos(3t - 19.4°)\text{ A} \qquad \text{and} \qquad i_3 = 0.6240\cos(3t + 33.7°)\text{ A}$$

Express the inductor voltage, v_2, as a function of time, t.

Plan:
This example is very similar to Example 2. First, apply KCL at the top node of the inductor to determine the inductor current, i_2, as a function of time, *t*. Next, use the element equation of the inductor to determine the inductor voltage as a function of time, *t*.

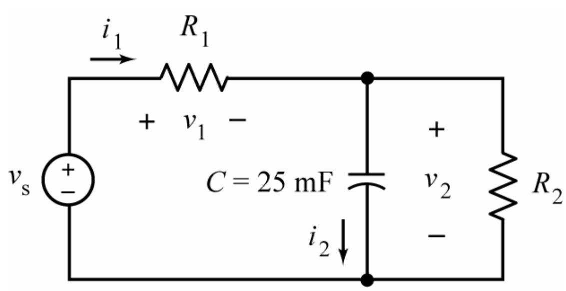
Solution:
First, apply Kirchhoff's current law (KCL) at the top node of the inductor to write

$$i_1 = i_2 - i_3 = 0 \quad \Rightarrow \quad i_2 = i_1 + i_3 = 1.040\cos(3t - 19.4°) + 0.6240\cos(3t + 33.7°)$$
$$= 0.8321\cos(3t - 56.3°) \text{ A}$$

Next, use the element equation of the inductor to write

$$v_2 = L\frac{d\,i_2}{d\,t} = 2\frac{d}{d\,t}0.8321\cos(3t - 56.3°) = 4.993\cos(3t + 33.7°) \text{ V}$$

Example 5:
Consider this circuit



where

$$v_s = \begin{cases} 10 \text{ V} & \text{for} \quad t < 0 \\ 20 \text{ V} & \text{for} \quad t > 0 \end{cases} \qquad \text{and} \qquad v_1 = \begin{cases} 2 \text{ V} & \text{for} \quad t < 0 \\ 8e^{-5t} + 4 \text{ V} & \text{for} \quad t > 0 \end{cases}$$

Given $R_1 = 10\ \Omega$, determine the value of R_2.

Plan:
This example is similar to Example 1. As in Example 1 we determine

$$v_2 = \begin{cases} 8 \text{ V} & \text{for} \quad t < 0 \\ 16 - 8e^{-5t} \text{ V} & \text{for} \quad t > 0 \end{cases} \qquad \text{and} \qquad i_2 = \begin{cases} 0 \text{ A} & \text{for} \quad t < 0 \\ 1e^{-5t} \text{ A} & \text{for} \quad t > 0 \end{cases}$$

Next, apply Ohm's law to resistor R_1 to determine i_1. Apply KCL at the top node of R_2 to determine the current in R_2. Noticing that v_2 is the voltage across R_2, apply Ohm's law to determine the value of R_2.

Solution:

Apply Ohm's law to resistor R_1 to write

$$i_1 = \frac{v_1}{R_1} = \frac{v_1}{10} = \begin{cases} 0.2 \text{ A} & \text{for} \quad t < 0 \\ 0.8e^{-5t} + 0.4 \text{ A} & \text{for} \quad t > 0 \end{cases}$$

Apply KCL at the top node of R_2 to write

$$\frac{v_2}{R_2} = i_1 - i_2 = \begin{cases} 0.2 \text{ A} & \text{for} \quad t < 0 \\ 0.4 - 0.2e^{-5t} \text{ A} & \text{for} \quad t > 0 \end{cases}$$

Comparing this equation to our equation for v_2, we conclude that $R_2 = 40\ \Omega$. (For example, when $t < 0$, $\frac{8}{R_2} = \frac{v_2}{R_2} = i_1 - i_2 = 0.2 \quad \Rightarrow \quad R_2 = 40\ \Omega$.)