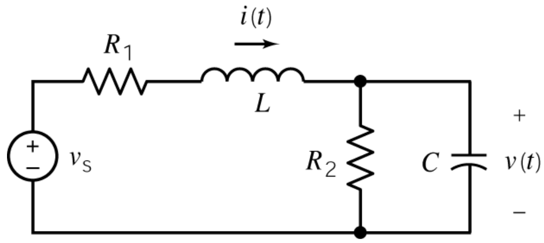
Example:



Represent this circuit by a second order differential equation.

Solution:
Use KCL to write

$$i(t)=\frac{v(t)}{R_2}+C\frac{d}{dt}v(t)$$

where $C\frac{d}{dt}v(t)$ is the current directed downward in the capacitor and $\frac{v(t)}{R_2}$ is the current directed downward in R_2. Use KVL to write

$$v_s = R_1\, i(t)+L\frac{d}{dt}i(t)+v(t)$$

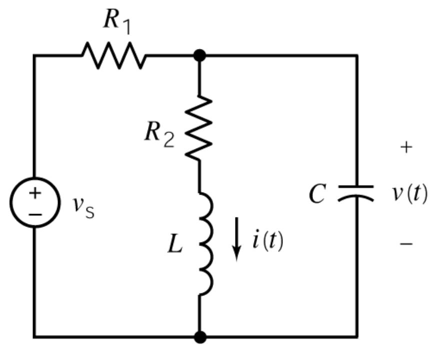
where $L\frac{d}{dt}i(t)$ is the voltage across the inductor and $R_1\, i(t)$ is the voltage across R_1.
Substitute to get

$$v_s=\frac{R_1}{R_2}v(t)+R_1C\frac{d}{dt}v(t)+\frac{L}{R_2}\frac{d}{dt}v(t)+CL\frac{d^2}{dt^2}v(t)+v(t)$$
$$=CL\frac{d^2}{dt^2}v(t)+\left(R_1C+\frac{1}{R_2C}\right)\frac{d}{dt}v(t)+\frac{R_1+R_2}{R_2CL}v(t)$$

Finally,

$$\frac{v_s}{CL}=\frac{d^2}{dt^2}v(t)+\left(\frac{R_1}{L}+\frac{1}{R_2C}\right)\frac{d}{dt}v(t)+\frac{R_1+R_2}{R_2CL}v(t)$$

Example:



Represent this circuit by a second order differential equation.

Solution:

Use KVL to get

$$R_2\, i(t) + L\frac{d}{dt}i(t) = v(t)$$

where $L\frac{d}{dt}i(t)$ is the voltage across the inductor and $R_2\, i(t)$ is the voltage across R_2. Use KCL and KVL to get

$$v_s = R_1\left(i(t) + C\frac{d}{dt}v(t)\right) + v(t)$$

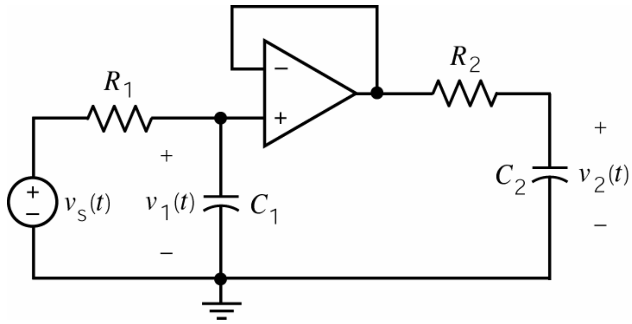
where $C\frac{d}{dt}v(t)$ is the current directed downward in the capacitor and $i(t)+C\frac{d}{dt}v(t)$ is the current directed to the left in R_1. Substitute to get

$$\begin{aligned} v_s &= R_1 v(t) + R_1 C R_2 \frac{d}{dt}i(t) + R_1 C\frac{d^2}{dt^2}i(t) + R_2 i(t) + L\frac{d}{dt}i(t) \\ &= R_1 CL\frac{d^2}{dt^2}i(t) + \left(R_1R_2 + L\right)\frac{d}{dt}i(t) + \left(R_1 + R_2\right)i(t) \end{aligned}$$

Finally

$$\frac{v_s}{R_1CL} = \frac{d^2}{dt^2}i(t) + \left(\frac{R_2}{L} + \frac{1}{R_1C}\right)\frac{d}{dt}i(t) + \frac{R_1+R_2}{R_1CL}i(t)$$

Example:
The input to the circuit this is the voltage of the voltage source, $v_s(t)$. The output is the voltage $v_2(t)$.



Derive the second order differential equation that shows how the output of this circuit is related to the input.

Solution:
KCL gives

$$\frac{v_s(t)-v_1(t)}{R_1}=C_1\frac{d}{dt}v_1(t) \quad \Rightarrow \quad v_s(t)=R_1C_1\frac{d}{dt}v_1(t)+v_1(t)$$

and

$$\frac{v_1(t)-v_2(t)}{R_2}=C_2\frac{d}{dt}v_2(t) \quad \Rightarrow \quad v_1(t)=R_2C_2\frac{d}{dt}v_2(t)+v_2(t)$$

Substituting gives

$$v_s(t)=R_1C_1\frac{d}{dt}\left[R_2C_2\frac{d}{dt}v_2(t)+v_2(t)\right]+R_2C_2\frac{d}{dt}v_2(t)+v_2(t)$$

so

$$\frac{1}{R_1R_2C_1C_2}v_s(t)=\frac{d^2}{dt^2}v_2(t)+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}\right)v_2(t)+\frac{1}{R_1R_2C_1C_2}v_2(t)$$