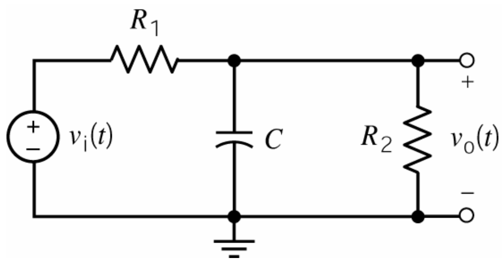
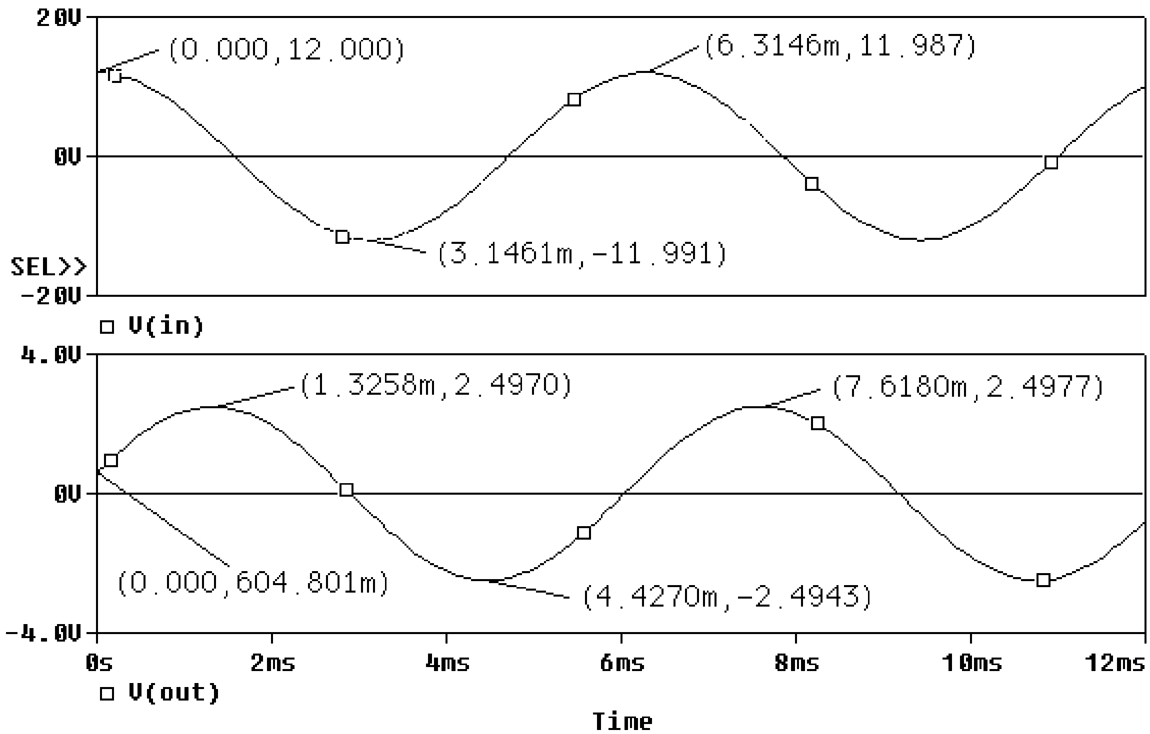
An Example involving an AC Circuit

The input to this circuit is the voltage source voltage, $v_i(t)$. The output is the voltage $v_o(t)$.



Here plots of $v_i(t)$ (top) and $v_o(t)$ (bottom) versus time:



Given $C = 0.2\ \mu\text{F}$, determine the values of R_1 and R_2:

$$R_1 = \_\_\_\_\_\_\_\_\ \Omega \text{ and } R_2 = \_\_\_\_\_\_\_\_\ \Omega.$$

Hint: Show that $v_i(t) = 12\cos(1000t)$ V and $v_o(t) = 2.5\cos(1000t - 76°)$ V.

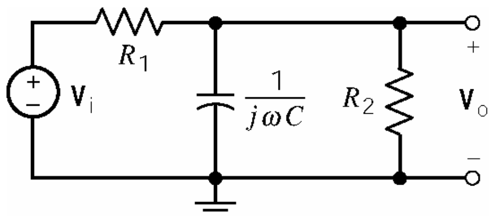
Solution:

From the top plot, the period of $v_i(t)$ is $T = 6.3146 \text{ ms} = 6.3146\times10^{-3} \text{ s}$. The frequency is $\omega = \frac{2\pi}{T} = \frac{2\pi}{6.3146\times10^{-3}} = 0.995\times10^{3} \cong 1000 \text{ rad/s}$. The amplitude of the input is 12 V and the phase angle is 0° so $v_i(t) = 12\cos(1000t) \text{ V}$.

From the bottom plot the amplitude of the output is $2.497 \cong 2.5$ and the phase angle is $-\cos^{-1}\left(\frac{0.6048}{2.5}\right) = -76°$. The input and output sinusoids have the same frequency so $v_o(t) = 2.5\cos(1000t - 76°) \text{ V}$.

Using phasors, we can summarize these results as $\frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = \frac{2.5\angle -76°}{12\angle 0°}$.

Next, we analyze the circuit in the frequency domain:



$$R_2 \,\Big\|\, \frac{1}{j\omega C} = \frac{R_2}{1 + j\omega CR_2}$$

$$\frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = \frac{\dfrac{R_2}{1 + j\omega CR_2}}{R_1 + \dfrac{R_2}{1 + j\omega CR_2}} = \frac{K}{1 + j\omega CR_p}$$

where $K = \frac{R_1}{R_1 + R_2}$ and $R_p = \frac{R_1 R_2}{R_1 + R_2}$

$$\frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = \frac{K}{\sqrt{1 + (\omega CR_p)^2}} e^{-j\tan^{-1}\omega CR_p}$$

Combining these results gives $\frac{2.5\angle -76°}{12\angle 0°} = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = \frac{K}{\sqrt{1 + (\omega CR_p)^2}} e^{-j\tan^{-1}\omega CR_p}$.

In this case the angle of $\frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)}$ is specified to be –76° so

$C R_p = C\frac{R_1 R_2}{R_1 + R_2} = -\frac{\tan(-76)}{1000} = 0.004$ and the magnitude of $\frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)}$ is specified to be $\frac{2.5}{12}$ so $\frac{K}{\sqrt{1+16}} = \frac{2.5}{12} \;\Rightarrow\; 0.859 = K = \frac{R_2}{R_1 + R_2}$. When $C = 0.2 \text{ μF}$ these equations indicate that $R_1 = 23.3 \text{ k}\Omega,\ R_2 = 142 \text{ k}\Omega$.

As a check, here's the PSpice circuit that produced the plots: