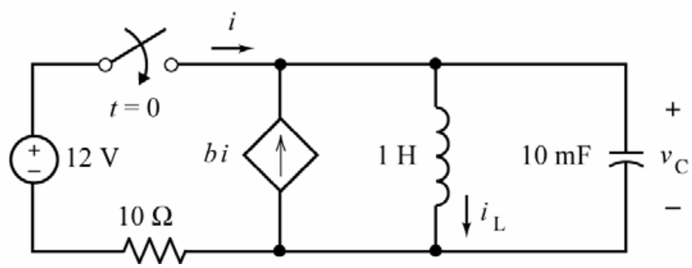
Example:
The initial conditions for this circuit are

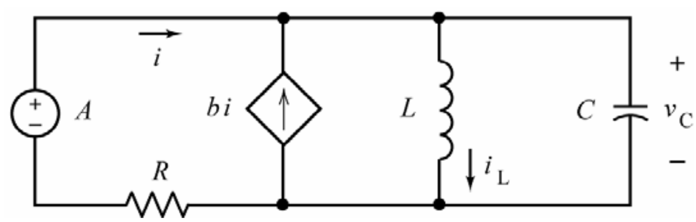
$v_C(0) = 0$ and $i_L(0) = 0$

Determine $v_C(t)$ for $t \geq 0$ in each of the following cases:

A. b = 1.5 A/A
B. b = 1 A/A
C. b = 0.2 A/A



Solution:
After the switch closes, we have



where A = 12 V, R = 10 Ω, L = 1 H and C = 0.01 F. Apply KVL to the outside loop to get

$$v_C + R\,i - A = 0 \quad \Rightarrow \quad i = \frac{A - v_C}{R}$$

Apply KCL at the bottom node of the indictor to get

$$i + b\,i = i_L + C\frac{d\,v_C}{dt} \quad \Rightarrow \quad (1+b)\frac{A - v_C}{R} = i_L + C\frac{d\,v_C}{dt} \qquad 1$$

Apply KVL to the mesh consisting of the capacitor and the indictor to get

$$L\frac{d\,i_L}{dt} = v_C \qquad 2$$

Notice that equation 2 indicates that

$$\left.\frac{d\,i_L}{dt}\right|_{t=0} = \frac{v_C(0)}{L} = 0$$

Substituting Equation 2 into Equation 1 gives

$$\frac{(1+b)}{R}A = C\,L\frac{d^2\,i_L}{dt^2} + (1+b)\frac{L}{R}\frac{d\,i_L}{dt} + i_L$$

Consequently, the circuit is represented by the second order differential equation

$$\frac{(1+b)}{C\,L\,R}A = \frac{d^2\,i_L}{dt^2} + \frac{(1+b)}{C\,R}\frac{d\,i_L}{dt} + \frac{i_L}{C\,L} \qquad 3$$

Take the Laplace transform of equation 3 to get

$$\frac{(1+b)}{C\,L\,R}\frac{A}{s} - = s^2\,I_L + \frac{(1+b)}{C\,R}s\,I_L + I_L$$

Solving for I_L gives

$$I_L = \frac{\frac{(1+b)}{C\,L\,R}A}{s\left(s^2 + \frac{(1+b)}{C\,R}s + 1\right)} = \frac{120(1+b)}{s\left(s^2 + 10(1+b)s + 100\right)}$$

A. For b = 1.5 A/A we have

$$I_L = \frac{300}{s\left(s^2 + 25\,s + 100\right)} = \frac{300}{s(s+5)(s+20)} = \frac{3}{s} - \frac{4}{s+5} + \frac{1}{s+5}$$

Taking the inverse Laplace transform gives

$$i_L = \left(3 - 4\,e^{-5t} + e^{-20t}\right)u(t)$$

Using equation 2

$$v_C = \frac{d\,i_L}{dt} = 20\left(e^{-5t} - e^{-20t}\right)u(t) + \left(3 - 4\,e^{-5t} + e^{-20t}\right)\delta(t)$$

The second term on the right side of this equation is always zero. It’s zero when $t = 0$ because the expression inside the parenthesis is evaluates to zero when $t = 0$. It’s zero at all other times because $\delta(t)$ is zero when $t \neq 0$. Consequently

$$v_C = \frac{d\,i_L}{dt} = 20\left(e^{-5t} - e^{-20t}\right)u(t)$$

Alternate Solution:

Take the Laplace transform of equations 1 and 2 to get

$$\frac{(1+b)}{R}\left(\frac{A}{s} - V_C\right) = I_L + C\left(sV_C - v_C(0)\right) = I_L + C\,sV_C$$

and

$$V_C = L\left(s\,I_L - i_L(0)\right) \quad \Rightarrow \quad V_C = L\,s\,I_L$$

Combining these equations we get

$$\frac{(1+b)}{R}\left(\frac{A}{s}-V_C\right)=\frac{V_C}{Ls}+C\,sV_C \quad\Rightarrow\quad \frac{(1+b)}{R}\left(\frac{A}{s}\right)Ls=V_C+\frac{(1+b)}{R}LsV_C+CLs^2V_C$$

Solving for V_C gives

$$V_C=\frac{\frac{A(1+b)}{RC}}{s^2+\frac{(1+b)}{RC}s+\frac{1}{CL}}=\frac{120(1+b)}{s^2+10(1+b)s+100} \qquad 4$$

D. For b = 1.5 A/A we have

$$V_C=\frac{300}{s^2+25s+100}=\frac{300}{(s+5)(s+20)}=\frac{20}{(s+5)}-\frac{20}{(s+20)}$$

Taking the inverse Laplace transform gives

$$v_C=20\left(e^{-5t}-e^{-20t}\right)u(t)$$

B. For b = 1.0 A/A we have

$$V_C=\frac{240}{s^2+20s+100}=\frac{240}{(s+10)^2}$$

Taking the inverse Laplace transform gives

$$v_C=\mathcal{L}^{-1}\left[\frac{240}{(s+10)^2}\right]=240e^{-10t}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]=240\,t\,e^{-10t}\,u(t)$$

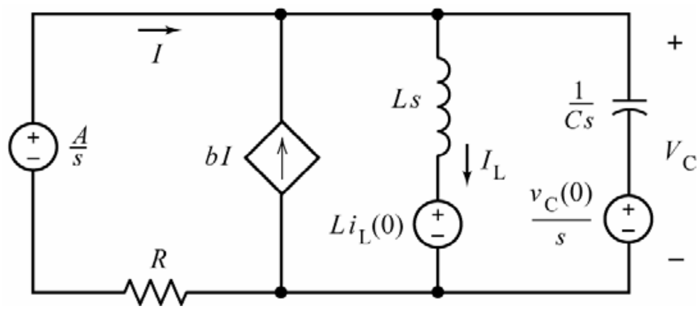
E. For b = 0.2 A/A we have

$$V_C=\frac{144}{s^2+12s+100}=\frac{144}{(s+6)^2+8^2}$$

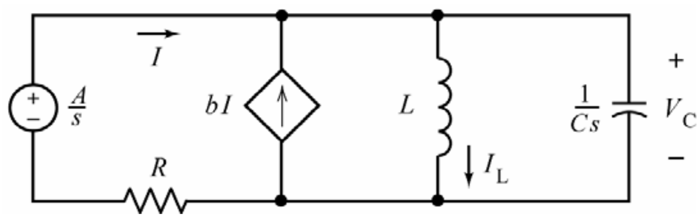
Taking the inverse Laplace transform gives

$$v_C=\mathcal{L}^{-1}\left[\frac{144}{(s+6)^2+8^2}\right]=18e^{-6t}\mathcal{L}^{-1}\left[\frac{8}{s^2+8^2}\right]=16e^{-6t}\sin(8t)u(t)$$

Second Alternate Solution:
Represent the circuit in the s-domain as



Because the initial conditions are zero, the voltage sources accounting for the initial conditions hare zero volt voltage sources and can be replaced by short circuits:



Apply KVL to the outside loop to get

$$V_C + RI - \frac{A}{s} = 0 \quad \Rightarrow \quad I = \frac{\frac{A}{s} - V_C}{R}$$

Apply KCL at the bottom node of the indictor to get

$$I + bI = I_L + \frac{V_C}{\frac{1}{Cs}} \quad \Rightarrow \quad \frac{(1+b)}{R}\frac{A}{s} = I_L + CsV_C + \frac{(1+b)}{R}V_C$$

Apply KVL to the mesh consisting of the capacitor and the indictor to get

$$LsI_L = V_C \quad \Rightarrow \quad I_L = \frac{V_C}{Ls}$$

Combining these equations and solving for V_C, we get

$$\frac{(1+b)}{R}\frac{A}{s} = \frac{V_C}{Ls} + CsV_C + \frac{(1+b)}{R}V_C \quad \Rightarrow \quad \frac{(1+b)A}{RC} = s^2 V_C + \frac{(1+b)}{RC}sV_C + \frac{V_C}{LC}$$

$$V_C = \frac{\frac{(1+b)A}{RC}}{s^2 + \frac{(1+b)}{RC}s + \frac{1}{LC}} = \frac{120(1+b)}{s^2 + 10(1+b)s + 100}$$

This is equation 4. Now v_C can be determined as before.

(checked using LNAP and MATLAB, 3/13/06)