The following examples illustrate the use of MATLAB for finding the inverse Laplace transform of functions having complex or repeated poles.

Example 1 *Repeated Real Poles*

Find the inverse Laplace transform of

$$V(s)=\frac{12}{s\left(s^2+8s+16\right)}$$

Solution

First, we will do this problem without using MATLAB. Noticing that $s^2+8s+16=(s+4)^2$, we begin the partial fraction expansion:

$$V(s)=\frac{12}{s\left(s^2+8s+16\right)}=\frac{12}{s(s+4)^2}=\frac{k}{s+4}+\frac{-3}{(s+4)^2}+\frac{\frac{3}{4}}{s}$$

Next, the constant k is evaluated by multiplying both sides of the last equation by $s(s+4)^2$.

$$12=ks(s+4)-3s+\frac{3}{4}(s+4)^2=\left(\frac{3}{4}+k\right)s^2+(3+4k)s+12 \quad\Rightarrow\quad k=-\frac{3}{4}$$

Finally $$v(t)=\mathfrak{L}^{-1}\left[\frac{-\frac{3}{4}}{s+4}+\frac{-3}{(s+4)^2}+\frac{\frac{3}{4}}{s}\right]=\left(\frac{3}{4}-e^{-4t}\left(\frac{3}{4}+3t\right)\right)u(t)\ \text{V}$$

Next, we perform the partial fraction expansion using the MATLAB function residue:

```
>>num = [12];
>>den = [1 8 16 0];
>>[r, p] = residue(num, den)
```

MATLAB responds

```
r =
   -0.7500
   -3.0000
    0.7500
p =
    -4
    -4
     0
```

A repeated pole of multiplicity m is listed m times corresponding to the m terms

$$\frac{k_1}{s-p},\ \frac{k_2}{(s-p)^2},\ \ldots,\ \frac{k_m}{(s-p)^m}$$

listed in order of increasing powers of $s-p$. The constants k_1, k_2,…,k_m are the corresponding residues, again listed in order of increasing powers of $s-p$. In our present case, the pole $p = -4$ has multiplicity 2 and the first two terms of the partial fraction expansion are

$$\frac{-0.75}{s-(-4)}+\frac{-3}{\left(s-(-4)\right)^2}=\frac{-0.75}{s+4}+\frac{-3}{(s+4)^2}$$

The entire partial fraction expansion is

$$\frac{-0.75}{s-(-4)}+\frac{-3}{\left(s-(-4)\right)^2}+\frac{0.75}{s-(0)}=\frac{-0.75}{s+4}+\frac{-3}{(s+4)^2}+\frac{0.75}{s}$$

Finally, as before

$$v(t)=\mathfrak{L}^{-1}\left[\frac{-0.75}{s+4}+\frac{-3}{(s+4)^2}+\frac{0.75}{s}\right]=\left(0.75-e^{-4t}\left(0.75+3t\right)\right)u(t)\ \text{ V}$$

Example 2 *Complex Poles*

Find the inverse Laplace transform of

$$V(s)=\frac{12s+78}{s^2+8s+52}$$

Solution

First, we will do this problem without using MATLAB. Notice that the denominator does not factor any further in the real numbers. Let's complete the square in the denominator

$$V(s)=\frac{12s+78}{s^2+8s+52}=\frac{12s+78}{\left(s^2+8s+16\right)+36}=\frac{12s+78}{(s+4)^2+36}=\frac{12(s+4)+30}{(s+4)^2+36}=\frac{12(s+4)}{(s+4)^2+6^2}+\frac{5(6)}{(s+4)^2+6^2}$$

Now use the property $e^{-at}f(t)\leftrightarrow F(s+a)$ and the Laplace transform pairs

$$\sin\omega t\ \text{ for } t\ge 0\leftrightarrow\frac{\omega}{s^2+\omega^2}\quad\text{and}\quad\cos\omega t\ \text{ for } t\ge 0\leftrightarrow\frac{s}{s^2+\omega^2}$$

to find the inverse Laplace transform:

$$v(t)=e^{-4t}\ \mathscr{L}^{-1}\left[\frac{12s}{s^2+6^2}+\frac{5(6)}{s^2+6^2}\right]=e^{-4t}\left[\,12\cos(6t)+5\sin(6t)\,\right]\ \text{ for } t>0$$

Next, we will use MATLAB to do the partial fraction expansion. First, enter the numerator and denominator polynomials as vectors listing the coefficients in order of decreasing power of s:

```
>>num = [12 78];
>>den = [1 8 52];
```

Now the command

```
>>[r, p] = residue(num, den)
```

tells MATLAB to do the partial fraction expansion return p, is a list of the poles of of $V(s)$, and r, a list of the corresponding residues. In the present case MATLAB returns

```
r =
   6.0000 - 2.5000i
   6.0000 + 2.5000i
p =
  -4.0000 + 6.0000i
  -4.0000 - 6.0000i
```

indicating $$V(s)=\frac{6-j2.5}{s-(-4+j6)}+\frac{6+j2.5}{s-(-4-j6)}$$

Notice that the first residue corresponds to the first pole and the second residue corresponds to the second pole. (Also, we expect complex poles to occur in pairs of complex conjugates and for the residues corresponding to complex conjugate poles to themselves be complex conjugates.) Taking the inverse Laplace transform, we get

$$v(t)=(6-j2.5)e^{-(-4+j6)t}+(6+j2.5)e^{-(-4-j6)t}$$

This expression, containing as it does complex numbers, isn't very convenient. Fortunately, we can use Euler's identity to obtain an equivalent expression that does not contain complex numbers. Since complex poles occur quite frequently, its worthwhile to consider the general case:

$$V(s)=\frac{a+jb}{s-(c+jd)}+\frac{a-jb}{s-(c-jd)}$$

The inverse Laplace transform is

$$\begin{aligned}v(t)&=(a+jb)e^{(c+jd)t}+(a-jb)e^{(c-jd)t}\\&=e^{ct}\left[(a+jb)e^{jdt}+(a-jb)e^{-jdt}\right]=e^{ct}\left[2a\left(\frac{e^{jdt}+e^{-jdt}}{2}\right)-2b\left(\frac{e^{jdt}-e^{-jdt}}{2j}\right)\right]\end{aligned}$$

Euler identity says $\frac{e^{jdt}+e^{-jdt}}{2}=\cos(dt)$ and $\frac{e^{jdt}-e^{-jdt}}{2j}=\sin(dt)$

Consequently $$v(t)=e^{ct}\left[2a\cos(dt)-2b\sin(dt)\right]$$

Thus we have the following Laplace transform pair

$$e^{ct}\left[2a\cos(dt)-2b\sin(dt)\right]\leftrightarrow\frac{a+jb}{s-(c+jd)}+\frac{a-jb}{s-(c-jd)}$$

In the present case $a = 6$, $b = -2.5$, $c = -4$ and $d = 6$ so we have

$$v(t)=e^{-4t}\left[12\cos(6t)+5\sin(6t)\right] \quad \text{for } t>0$$

It's sometimes convenient to express this answer in a different form. First, express the sine term as an equivalent cosine

$$v(t) = e^{-4t}\left[12\cos(6t) + 5\cos(6t - 90°)\right] \quad \text{for } t > 0$$

Next use phasors to combine the cosine terms

$$\mathbf{V}(\omega) = 12\angle 0° + 5\angle -90° = 12 - j5 = 13\angle -22.62°$$

Now $v(t)$ is expressed as $\quad v(t) = 13e^{-4t}\cos(6t - 22.62°) \quad \text{for } t > 0$

Example 3 *Both Real and Complex Poles*

Find the inverse Laplace transform of

$$V(s) = \frac{105s + 840}{(s^2 + 9.5s + 17.5)(s^2 + 8s + 80)} = \frac{105s + 840}{s^4 + 17.5s^3 + 173.5s^2 + 900s + 1400}$$

Solution

Using MATLAB

```
>> num=[105 840];
>> den=conv([1 9.5 17.5],[1 8 80]);
>> [r,p] = residue(num, den)
r =
  -0.8087 + 0.2415i
  -0.8087 - 0.2415i
  -0.3196
   1.9371
p =
  -4.0000 + 8.0000i
  -4.0000 - 8.0000i
  -7.0000
  -2.5000
```

Consequently

$$V(s) = \frac{-0.8087 + j0.2415}{s - (-4 + j8)} + \frac{-0.8087 - j0.2415}{s - (-4 - j8)} + \frac{-0.3196}{s - (-7)} + \frac{1.9371}{s - (-2.5)}$$

Using the Laplace transform pair

$$e^{ct}\left[2a\cos(dt) - 2b\sin(dt)\right] \leftrightarrow \frac{a + jb}{s - (c + jd)} + \frac{a - jb}{s - (c - jd)}$$

with $a = -0.8087$, $b = 0.2415$, $c = -4$ and $d = 8$ we have

$$\mathfrak{L}^{-1}\left[\frac{-0.8087 + j0.2415}{s - (-4 + j8)} + \frac{-0.8087 - j0.2415}{s - (-4 - j8)}\right] = e^{-4t}\left[-1.6174\cos(8t) + 0.483\sin(8t)\right]$$

Taking the inverse Laplace transform of the remaining terms of $V(s)$ we get

$$v(t) = e^{-4t}\left[-1.6174\cos(8t) + 0.483\sin(8t)\right] - 0.3196\,e^{-7t} + 1.9371\,e^{-2.5t} \quad \text{for } t > 0$$

Problem 1

Find the inverse Laplace transform of $V(s) = \dfrac{11.6s^2 + 91.83s + 186.525}{s^3 + 10.95s^2 + 35.525s + 29.25}$

Solution 1
Using MATLAB:

```
>> num = [11.6  91.83  186.525];
>> den = [1  10.95  35.525  29.25];
>> [r,p]=residue(num,den)
r =
    8.2000
   -3.6000
    7.0000
p =
   -5.2000
   -4.5000
   -1.2500
```

Consequently

$$V(s) = \frac{8.2}{s-(-5.2)} + \frac{-3.6}{s-(-4.5)} + \frac{7}{s-(-1.25)} = \frac{8.2}{s+5.2} + \frac{-3.6}{s+4.5} + \frac{7}{s+1.25}$$

and $$v(t) = 8.2\,e^{-5.2t} - 3.6\,e^{-4.5t} + 7\,e^{-1.25t} \quad \text{for } t > 0$$

Problem 2

Find the inverse Laplace transform of $V(s) = \dfrac{8s^3 + 139s^2 + 774s + 1471}{s^4 + 12s^3 + 77s^2 + 296s + 464}$

Solution 2
Using MATLAB:

```
>> num = [8 139 774 1471];
>> den = [1 12 77 296 464];
>> [r,p]=residue(num,den)
r =
   3.0000 - 6.0000i
   3.0000 + 6.0000i
   2.0000
   3.0000
p =
  -2.0000 + 5.0000i
  -2.0000 - 5.0000i
  -4.0000
  -4.0000
```

Consequently $$V(s)=\frac{3-j6}{s-(-2+j5)}+\frac{3+j6}{s-(-2-j5)}+\frac{2}{s-(-4)}+\frac{3}{(s-(-4))^2}$$

Using the Laplace transform pair

$$e^{ct}\left[2a\cos(dt)-2b\sin(dt)\right]\leftrightarrow\frac{a+jb}{s-(c+jd)}+\frac{a-jb}{s-(c-jd)}$$

with $a = 3$, $b = -6$, $c = -2$ and $d = 5$ we have

$$v(t)=e^{-2t}\left(6\cos(5t)+12\sin(5t)\right)+e^{-4t}(2+3t) \quad \text{for } t>0$$

Problem 3

Find the inverse Laplace transform of $V(s)=\dfrac{s^2+6s+11}{s^3+12s^2+48s+64}=\dfrac{s^2+6s+11}{(s+4)^3}$

Solution 3

Using MATLAB:

```
>> num = [1 6 11];
>> den = [1 12 48 64];
>> [r,p]=residue(num,den)
r =
    1.0000
   -2.0000
    3.0000
p =
   -4.0000
   -4.0000
   -4.0000
```

Consequently $$V(s)=\frac{1}{s-(-4)}+\frac{-2}{(s-(-4))^2}+\frac{3}{(s-(-4))^3}$$

and $$v(t)=e^{-4t}\left(1-2t+3t^2\right) \quad \text{for } t>0$$

Problem 4

Find the inverse Laplace transform of $V(s)=\dfrac{-60}{s^2+5s+48.5}$

Solution 4

The denominator does not factor any further in the real numbers. Let's complete the square in the denominator

$$V(s)=\frac{-60}{s^2+5s+48.5}=\frac{-60}{(s+2.5)^2+42.25}=\frac{-60}{(s+2.5)^2+6.5^2}=\frac{-9.23(6.5)}{(s+2.5)^2+6.5^2}$$

Now use $e^{-at} f(t) \leftrightarrow F(s+a)$ and sin ωt for $t > 0 \leftrightarrow \frac{\omega}{s^2+\omega^2}$ to find the inverse Laplace transform

$$v(t) = e^{-2.5t} \ \mathscr{L}^{-1}\left[\frac{-9.23(6.5)}{(s+2.5)^2+6.5^2}\right] = -9.23 \ e^{-2.5t} \ \sin(6.5\ t) \text{ for } t > 0$$

Using MATLAB:

```
>> num = [-60];
>> den = [1 5 48.5];
>> [r,p]=residue(num,den)
r =
        0 + 4.6154i
        0 - 4.6154i
p =
  -2.5000 + 6.5000i
  -2.5000 - 6.5000i
```

Using the Laplace transform pair

$$e^{ct}\left[2a\cos(d\,t) - 2b\sin(d\,t)\right] \leftrightarrow \frac{a+jb}{s-(c+jd)} + \frac{a-jb}{s-(c-jd)}$$

with $a = 0$, $b = 4.6154$, $c = -2.5$ and $d = 6.5$ we have

$$v(t) = -9.2308\,e^{-2.5t}\sin(6.5t) \quad \text{for } t > 0$$

Problem 5

Find the inverse Laplace transform of $V(s) = \frac{-30}{s^2-25}$

Solution 5

Using MATLAB:

```
>> num = [-30];
>> den = [1  -25];
>> [r,p]=residue(num,den)
r =
    -3
     3
p =
     5
    -5
```

Consequently $$V(s) = \frac{-3}{s-5} + \frac{3}{s-(-5)} = \frac{-3}{s-5} + \frac{3}{s+5}$$

and $$v(t) = -3e^{5t} + 3e^{-5t} \quad \text{for } t > 0$$