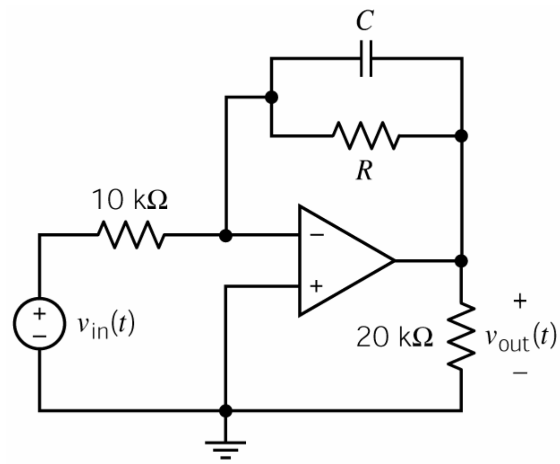
Circuits and the Fourier Series

Example1:



The input to this circuit is the voltage of the voltage source

$$v_{\text{in}}(t) = 2 + 4\cos(100t) + 5\cos(400t + 45°) \text{ V}$$

The output is the voltage across the 20-kΩ resistor

$$v_{\text{out}}(t) = -5 + 7.071\cos(100t + 135°) + c_4\cos(400t + \theta_4) \text{ V}$$

Determine the values of the resistance, *R*, the capacitance, *C*, the coefficient, c_4, and the phase angle, θ_4.

Solution:

The network function of the circuit is:

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = -\frac{R \parallel \dfrac{1}{j\omega C}}{10^4} = -\frac{\dfrac{R}{1 + j\omega CR}}{10^4} = -\frac{\dfrac{R}{10^4}}{1 + j\omega CR} = \frac{\dfrac{R}{10^4}}{\sqrt{1 + (\omega CR)^2}} \angle\left(180 - \tan^{-1}(\omega CR)\right)$$

The phasors of the input and output are related by

$$|\mathbf{V_o}(\omega)||\mathbf{H}(\omega)| = |\mathbf{V_i}(\omega)| \quad \text{and} \quad \angle\mathbf{V_o}(\omega) = \angle\mathbf{H}(\omega) + \angle\mathbf{V_i}(\omega)$$

When $\omega = 0$ (dc)

$$\frac{-5}{2} = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = -\frac{\dfrac{R}{10^4}}{1 + j(0)CR} = -\frac{R}{10^4} \quad \Rightarrow \quad R = 25 \text{ k}\Omega$$

When ω = 100 rad/s

$$135° = \angle \mathbf{H}(\omega) + 0° = 180° - \tan^{-1}(\omega C R) \quad \Rightarrow \quad \tan(45°) = (100)C(25000)$$
$$\Rightarrow \quad C = 0.4 \ \mu\text{F}$$

When ω = 100 rad/s

$$c_4 = (5)\left|\mathbf{H}(400)\right| = (5)\left|\frac{\frac{25000}{10^4}}{1 + j(400)(0.4\times10^{-6})(25000)}\right| = 3.032$$

$$\theta_4 = 45° + \angle \mathbf{H}(400) = 45° + 180 - \tan^{-1}\left(400 \times 0.4\times10^{-6} \times 25000\right) = 149°$$

Example 2:
The input to a circuit is the voltage

$$v_{\text{i}}(t) = 2 + 4\cos(25t) + 5\cos(100t + 45°) \ \text{ V}$$

The output is the voltage

$$v_{\text{o}}(t) = 5 + 7.071\cos(25t - 45°) + c_4\cos(100t + \theta_4) \ \text{ V}$$

The network function that represents this circuit is

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = \frac{H_{\text{o}}}{1 + j\frac{\omega}{p}}$$

Determine the values of the dc gain, H_{o}, the pole, p, the coefficient, c_4, and the phase angle, θ_4.

Solution:
The transfer function is

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = \frac{H_{\text{o}}}{1 + j\frac{\omega}{p}} = \frac{H_{\text{o}}}{\sqrt{1 + \left(\frac{\omega}{p}\right)^2}} \angle\left(\angle H_{\text{o}} - \tan^{-1}\left(\frac{\omega}{p}\right)\right)$$

The phasors of the input and output are related by

$$\left|\mathbf{V_o}(\omega)\right|\left|\mathbf{H}(\omega)\right| = \left|\mathbf{V_i}(\omega)\right| \ \text{ and } \ \angle\mathbf{V_o}(\omega) = \angle\mathbf{H}(\omega) + \angle\mathbf{V_i}(\omega)$$

When ω = 0 (dc)

$$5=\frac{H_o}{1+j\frac{0}{p}}(2)=H_o(2) \quad\Rightarrow\quad H_o=2.5 \text{ V/V}$$

Consequently, $\angle\mathbf{H}(\omega)=\angle H_o-\tan^{-1}\left(\frac{\omega}{p}\right)=0-\tan^{-1}\left(\frac{\omega}{p}\right)=-\tan^{-1}\left(\frac{\omega}{p}\right)$

When ω = 25 rad/s

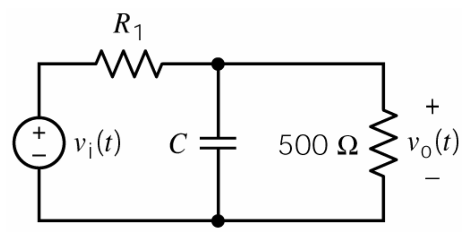
$$-45°=\angle\mathbf{H}(\omega)+0=-\tan^{-1}\left(\frac{\omega}{p}\right) \quad\Rightarrow\quad \tan(45°)=\frac{25}{p} \quad\Rightarrow\quad p=25 \text{ rad/s}$$

When ω = 100 rad/s

$$c_4=(5)\left|\mathbf{H}(100)\right|=(5)\left|\frac{2.5}{1+j\frac{100}{25}}\right|=3.03$$

$$\theta_4=45°+\angle\mathbf{H}(100)=45°-\tan^{-1}\left(\frac{100}{25}\right)=-31°$$

Example:



The input to this circuit is the voltage of the independent voltage source

$$v_i(t)=6+4\cos(1000t)+5\cos(3000t+45°) \text{ V}$$

The output is the voltage across a 500-Ω resistor

$$v_o(t)=3.75+2.34\cos(1000t-20.5°)+c_3\cos(3000t+\theta_3) \text{ V}$$

Determine the values of the resistance, R_1, the capacitance, C, the coefficient, c_3, and the phase angle, θ_3.

Solution:
The network function of this circuit

$$\mathbf{H}(\omega)=\frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)}=\frac{R_2 \,||\, \dfrac{1}{j\omega C}}{R_1+R_2 \,||\, \dfrac{1}{j\omega C}}=\frac{R_2}{R_1+R_2+j\omega C\, R_1\, R_2}$$

where $R_2=500\ \Omega$.

The phasors of the input and output are related by

$$\left|\mathbf{V_o}(\omega)\right|\left|\mathbf{H}(\omega)\right|=\left|\mathbf{V_i}(\omega)\right| \text{ and } \angle\mathbf{V_o}(\omega)=\angle\mathbf{H}(\omega)+\angle\mathbf{V_i}(\omega)$$

When ω = 0 (dc)

$$\frac{R_2}{R_1+R_2}=\frac{3.75}{6} \quad\Rightarrow\quad R_1=\left(\frac{2.25}{3.75}\right)R_2=\left(\frac{2.25}{3.75}\right)(500)=300\ \Omega$$

When ω = 1000 rad/s

$$-20.5°=\angle\mathbf{H}(\omega)=-\tan^{-1}\left(\omega C\frac{R_1\, R_2}{R_1+R_2}\right) \quad\Rightarrow\quad \tan(20.5°)=(1000)C\left(\frac{(300)(500)}{800}\right)$$

$$\Rightarrow\quad C=2\ \ \mu\text{F}$$

When ω = 3000 rad/s

$$c_3\angle\theta_3=\left(\frac{500}{800+j(3000)(2\times10^{-6})(500)(300)}\right)(5\angle45°)=2.076\angle-3.4°$$