# Stochastic synchronization over a moving neighborhood network

Maurizio Porfiri
Department of Mechanical, Aerospace, and Manufacturing Engineering
Polytechnic University
Brooklyn, NY 11201
*mporfiri@poly.edu*

Daniel J. Stilwell
The Bradley Department of Electrical and Computer Engineering
Virginia Polytechnic Inst. and State Univ.
Blacksburg, VA 24061
*stilwell@vt.edu*

Erik M. Bollt    Joseph D. Skufca
Department of Mathematics and Computer Science
Clarkson University
Postdam, NY 13699-5815
{*bolltem,jskufca*}*@clarkson.edu*

*Abstract*— **We examine the synchronization problem for a group of dynamic agents that communicate via a moving neighborhood network. Each agent is modeled as a random walker in a finite lattice and is equipped with an oscillator. The communication network topology changes randomly and is dictated by the agents' locations in the lattice. Information sharing (talking) is possible only for geographically neighboring agents. The complex system is a time-varying jump nonlinear system. We introduce the concept of long-time expected communication network defined as the ergodic limit of the stochastic time-varying network. We show that if the long-time expected network supports synchronization, then so does the stochastic network when the agents diffuse sufficiently fast in the lattice.**

*Keywords:* synchronization, random walk, stochastic stability, graph, fast switching

## I. Introduction

Over the past few years, synchronization in complex networks has attracted a massive research attention, see the excellent reviews [1], [2]. Synchronization problems can be found in wide variety of phenomena, ranging from epidemics [3], to biological systems [4], sociology [5], chemistry [6], nonlinear optics [7], and meteorology [8].

Despite the very large literature to be found, the great majority of research activities have been focused on static networks whose connectivity and coupling strengths are constant in time. For example, static networks are assumed for the analysis of [9], [10], [11], [12], [13], [14], [15], [16]. However, in many real-world complex networks, such as biological, epidemiological and social networks, it is reasonable to assume that the coupling strengths and even the network topology can evolve in time. Recent works such as [17], [18], [19], [20], [21], [22], [23] are amongst the few to consider time-dependent couplings.

To our knowledge, [22] is the first research attempt, in the synchronization literature, to naturally model the time evolution of the communication network. In particular, in [22] synchronization over a network of diffusing agents communicating within geographical neighborhoods is considered. Each agent carries an oscillator and moves ergodically in the environment. A communication network is formed based on the agents' motion. When two diffusing agents are close, a communication link is established and information sharing is possible: the agents talk. When the same two agents move apart, the communication link is removed, and information sharing is prohibited. Through numerical experiments it is shown that the oscillators' synchronization is possible even if the communication network is mostly disconnected at any frozen time instants. It is conjectured that synchronization of the set of oscillators can be assessed by examining a time-averaged communication network that is computed from the underlying time-varying and sparsely connected network. The model proposed in [22] seems particularly promising for modeling social interactions and epidemic spreading.

In [23] we made a first attempt to mathematically formalize our results of [22] . However, whereas in [22], our agents moved ergodically in an underlying state space, which automatically formed time varying networks, our work in [23] simply concerned deterministic switching between a finite collection of networks representing couplings between oscillators. Network switching was not due to agents' motion. In this paper, for the first time we are able to put the results in [22] onto a much firmer footing, using a related system, where we assume agents randomly walking through an underlying graph.

We consider a system of $N$ identical agents that meander in a finite region. Each agent carries an oscillator and diffuses in the environment as a random walker. Each agent is described in terms of a spatial coordinate $X$, specifying its random walk, and a state variable $x$, characterizing the oscillator's state. We assume that the random walkers are independent and that diffusion takes place in a bounded lattice described by a finite, connected and not bipartite graph. In addition we assume that the random walkers move only at prescribed instants in time, equally spaced with a period $\Delta$. The period $\Delta$ is a measure of the time-scale of the diffusion process, the smaller $\Delta$ is, the faster the agents meander in the environment.

We associate to the random walks a time-varying communication network, represented by a graph on a $N$-dimensional vertex set that we name the *moving neighborhood graph*. The edges of the moving neighborhood graph are determined by the agents locations in the lattice, that is a link between two agents is present only when the corresponding random walkers occupy the same site of the lattice. Therefore, this graph is generally not connected at frozen instants in time. The time evolution of the moving neighborhood graph, due to motion of the agents in the lattice, is called the *network dynamics* and is independent by the states of the oscillators. When a link is present between two agents in the moving neighborhood graph, the corresponding oscillator systems are dynamically coupled. The time evolution of the set of oscillators is called the *system dynamics* and is influenced by the network dynamics.

The set of oscillators coupled by the moving neighborhood graph are synchronized if all their states are equal. We determine sufficient conditions for asymptotic synchronization by combining results from Markov chains, stochastic stability and fast switching theory. We define the long-time expected communication network as the ergodic limit of the moving neighborhood graph. In the long-time average, the network behaves like an all-to-all coupling scheme among the oscillators and the related synchronization problem may be addressed by using the well-known master stability function, see e.g. [9], [10], [13]. We show that if the oscillators synchronize when coupled by the all-to-all network, then synchronization is possible if the period $\Delta$ is sufficiently small.

## II. Review of relevant terms

### A. Markov Chains

A sequence of discrete valued random variables $X(k)$, $k \in \mathbb{Z}^+$ with sample space $F$ is called a *Markov Chain* if it satisfies the *Markov Condition*

$$\mathsf{P}(X(k+1) = s | X(k) = x_k, X(k-1) = x_{k-1}, \dots X(1) = x_1) = \mathsf{P}(X(k+1) = s | X(k) = x_k)$$

for all $k \in \mathbb{Z}^+$ and all $s, x_1, \dots, x_k \in F$. In this paper, we specialize to *homogeneous Markov chains,* which implies the additional property that

$$\mathsf{P}(X(k+1) = j | X(k) = i) = \mathsf{P}(X(2) = j | X(1) = i), \quad \forall k \in \mathbb{Z}^+, \text{ and } i, j \in F$$

Additional, we assume that $F$ is finite and we indicate with $|F|$ its cardinality. Without loss of generality, we number the possible states of the chain using positive integers so that $F = \{1, \dots |F|\}$. (For ease of description, when we say *Markov chain* in this paper, we are considering only the restriction to this smaller set of finite, homogeneous processes.)

The matrix $P = [p_{ij}]$ where $p_{ij} = \mathsf{P}(X(k+1) = j | X(k) = i)$ is called the transition matrix of the Markov chain and is a stochastic matrix, that is, it is nonnegative and all its rows sum to one. A matrix is nonnegative (positive) if all its entries are greater or equal (greater) than zero. Since the rows of $P$ sums to one, the $|F|-$vector $e_{|F|} = [1, \dots 1]^{\mathrm{T}}$ is always an eigenvector of $P$ corresponding to an eigenvalue equal to one. The random variable $X(0)$ is called the initial state of the Markov chain and its probability distribution $\pi(0) = [\pi_1(0), \dots \pi_{|F|}(0)]^{\mathrm{T}}$, defined as

$$\pi_i(0) = \mathsf{P}(X(0) = i)$$

is called the initial distribution. The distribution of the chain at the $k^{th}$ time step $\pi(k) = [\pi_1(k), \dots \pi_{|F|}(k)]^{\mathrm{T}}$ is defined as

$$\pi_i(k) = \mathsf{P}(X(k) = i)$$

and can be expressed in terms of the initial distribution and the transition matrix by using the Chapman-Kolmogorov equation, see e.g. Theorem 1.1 in Chapter 2 of [24], as

$$\pi_r(k)^{\mathrm{T}} = \pi_r(0)^{\mathrm{T}} P^k \tag{1}$$

Two states $i$ and $j$ are said to communicate if there exists $\bar{k} \in \mathbb{Z}^+$ such that the $ij$ and $ji$ entries of $P^{\bar{k}}$, say $p_{ij}^{(\bar{k})}$ and $p_{ji}^{(\bar{k})}$, are positive. If all the states in $F$ communicate the chain is called irreducible and $P$ is an irreducible matrix. The period $d_i$ of a state $i$ is defined by $d_i = \gcd\{n \in \mathbb{Z}^+ \setminus \{0\} : p_{ii}^{(n)} > 0\}$, and if $d_i = 1$ the state $i$ is called aperiodic. For an irreducible Markov chain all the states have the same period, see e.g. Section 2.4 in [24]. An irreducible aperiodic Markov chain is called ergodic. The transition matrix of an ergodic Markov chain is a primitive stochastic matrix, see e.g. Chapter 6 of [24]. $P$ being primitive means that there exists $\bar{k} \in \mathbb{Z}^+$ such that $P^{\bar{k}}$ is a positive matrix. A probability distribution $\pi$ is called stationary if $\pi^{\mathrm{T}} = \pi^{\mathrm{T}} P$. If all the entries of $\pi$ are positive and $\pi_i p_{ij} = \pi_j p_{ji}$ for all the states $i$ and $j$, the Markov chain is called reversible. The spectrum of the transition matrix of an ergodic Markov chain is $\lambda_1, \dots \lambda_{|F|}$, with $\lambda_1 = 1$ having algebraic and geometric multiplicity equal to one and with $|\lambda_r| < 1$ for $r = 2, \dots |F|$. For a reversible Markov chain all the eigenvalues of $P$ are real. For an ergodic Markov chain, there exist a positive constants $\mu$ and a positive constant $\rho < 1$ such that

$$|p_{ij}^{(k)} - \pi_j| \leq \mu \rho^k \tag{2}$$

where $p_{ij}^{(k)}$ is the $ij$ entry of $P^k$, and $\pi$ is the unique stationary distribution of the chain, see e.g. [25]. From (2), for any initial probability distribution and state $j \in F$ the distribution at the $k^{th}$ time step satisfy the ergodicity condition

$$|\pi_j(k) - \pi_j| \leq \mu \rho^k \tag{3}$$

### B. Graphs

A graph is a pair of sets $G = (E, V)$, where $V = \{1, \dots |V|\}$ and $E \subseteq V \times V$ are finite, see e.g. [26]. The elements of $V$ are called nodes, vertices or sites, and the elements of $E$ are *unordered pairs* and are called edges or links. Two nodes $q, r \in V$ are neighbors if there is an edge connecting them, that is if the *unordered pair* $(q, r) \in E$. A path from $q$ to $r$ is a sequence of distinct vertices starting with $q$ and ending with $r$ such that the consecutive vertices are neighbors. The graph $G$ is connected if there exists a path between every two vertices in $V$. If the graph is connected, all the components of $d$ are nonzero. The graph is called bipartite if $V$ can be partitioned into two subsets $V_1$ and $V_2$, such that every edge has one end in $V_1$ and the other in $V_2$. A graph is said to have a self-loop at node $q$ if $(q, q) \in E$. The graph topology can be algebraically represented by introducing the adjacency matrix $A = [a_{qr}]$ defined by

$$a_{qr} = \begin{cases} 1 & \text{if} \quad (q,r) \in E \\ 0 & \text{otherwise} \end{cases}$$

Since edges are represented by unordered pairs, we may immediately infer that $A$ is symmetric. The degree matrix $D = \mathrm{diag}(d)$ is a diagonal matrix, whose diagonal elements are $d_q = \sum_{r=1}^{n} a_{qr}$. The Laplacian matrix $L = [l_{qr}]$ is defined as the difference between the adjacency matrix and the degree matrix, that is $L = D - A$. The graph Laplacian is a symmetric positive semidefinite matrix. Spectral properties of graph Laplacians may be found for example in [27] and [28].

The Laplacian matrix is a zero row-sum matrix. Therefore the null space of $L$ contains the $|V|-$vector $e_{|V|} = [1, \dots 1]^{\mathrm{T}}$ corresponding to the zero eigenvalue. The multiplicity of the zero eigenvalue is one if and only if the graph is connected. The highest eigenvalue is less or equal to $\max\{d_q + d_r : (q,r) \in E\}$ (see e.g. [28]), which is less than $2|V|$.

## III. Problem Statement

### A. *Moving Neighborhood Network*

Let the network dynamics be described by a set of $N$ independent random walkers $X_1, \dots X_N$ on a finite not bipartite connected graph $G^{\text{rw}} = (E^{\text{rw}}, V^{\text{rw}})$, see e.g. [29]. (We use the superscript $(\cdot)^{\text{rw}}$ to denote that these objects are associated with lattice graph that determines where the random walkers can move.) Each random walk represents the motion of an agent in the system. The initial probability distribution of each random walker $X_q$, $q = 1, \dots N$, is $\pi_q(0) = [\pi_{q1}(0), \dots \pi_{q|V^{\text{rw}}|}(0)]^{\mathrm{T}}$.

Consider a random walker $X_q$ on $G^{\text{rw}}$. If at the $k^{th}$ time step, $k \in \mathbb{Z}^+$, $X_q$ is located at the site $i$, that is $X_q(k) = i$, we allow the walker to move to any of its neighboring sites with equal probability, so that

$$p_{ij} = \begin{cases} 1/d_i^{\text{rw}} & (i,j) \in E^{\text{rw}} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The sequence of positions $X_q(k)$ is a Markov chain on the set $V^{\text{rw}}$ with transition matrix $P = [p_{ij}]$. Since the graph is connected and not bipartite, the Markov chain is ergodic, see e.g. [29]. The stationary probability distribution is

$$\pi_i = \frac{d_i^{\text{rw}}}{\sum_{j=1}^{|V^{\text{rw}}|} d_j^{\text{rw}}} \tag{5}$$

and the chain is reversible.

The random walkers move independently from each other, and share the same transition matrix $P$ and stationary distribution $\pi$. From (3), for any initial probability distribution and site $j \in V^{\text{rw}}$ the distribution at the $k^{th}$ time step satisfies (3).

The overall state of the independent random walkers may be represented by a sole augmented Markov chain on a space of cardinality $|V^{\text{rw}}|^N$ whose transition matrix is $P \otimes \dots \otimes P$ ($N$ times), where '$\otimes$' is the standard Kronecker product.

Given that the static graph $G^{\text{rw}}$ describes the lattice where the agents meander, we now introduce a second graph $G^{\text{mn}}$ that describes the agents' talking. The moving neighborhood graph $G^{\text{mn}}(k) = (E^{\text{mn}}(k), V^{\text{mn}})$ is a sequence of random graphs whose node set is $V^{\text{mn}} = \{1, \dots N\}$, where we associate a node with each of the random walkers, and whose edges depend on the random walkers' location in the graph $G^{\text{rw}}$. (Note that for the moving neighborhood graph, we use superscripts $(\cdot)^{\text{mn}}$ to distinguish this communication graph from the lattice where the random walkers are moving.) In the $k^{th}$ time interval, the set of edges of $G^{\text{mn}}(k)$ is defined by

$$E^{\text{mn}}(k) = \{(q,r) \in V^{\text{mn}} \times V^{\text{mn}}, q \neq r : X_q(k) = X_r(k)\}$$

that is the edge $(q,r)$ is present at the $k^{th}$ time step if and only if the random walkers $X_q$ and $X_r$ occupy the same site in the graph $G^{\text{rw}}$ during the $k^{th}$ step. Clearly, the moving neighborhood graph does not have self-loops.

For $q \neq r$, The $qr$ entry of the expected value of the adjacency matrix of the moving neighborhood graph $G^{\text{mn}}$ at the $k^{th}$ step is

$$\mathsf{E}[a_{qr}^{\text{mn}}(k)] = \sum_{i=1}^{|V^{\text{rw}}|} \pi_{qi}(k)\pi_{ri}(k) \tag{6}$$

and it represents the probability that the $q^{th}$ and $r^{th}$ random walkers occupy the same site in the graph $G^{\text{rw}}$ during the $k^{th}$ time interval. The expected value of the $r^{th}$ diagonal element of the degree matrix at the the $k^{th}$ time interval is

$$\mathsf{E}[d_q^{\text{mn}}(k)] = \sum_{r=1, r\neq q}^{N} \sum_{i=1}^{|V^{\text{rw}}|} \pi_{qi}(k)\pi_{ri}(k) \tag{7}$$

and it represents the probability that the $q^{th}$ random walker occupies the same site of any other random walker in the graph $G^{\text{rw}}$ during the $k^{th}$ time interval. Therefore, the expected value of the graph Laplacian in the $k^{th}$ time interval is $\mathsf{E}[L^{\text{mn}}(k)] = \mathsf{E}[D^{\text{mn}}(k)] - \mathsf{E}[A^{\text{mn}}(k)]$, and it is a zero-row sum matrix.

We have a notion of the long-time expected graph which describes the communication of the agents with respect to the stationary distribution $\pi$. We note that the sequence of random graphs $G^{\text{mn}}(k)$ is used to generate several sequences of random variables, such as $A^{\text{mn}}(k)$ and $L^{\text{mn}}(k)$. For a sequence of random variables $Y(k)$, we introduce the ergodic limit, $\mathsf{E}^*[Y]$, defined by

$$\mathsf{E}^*[Y] = \lim_{k\to\infty} \mathsf{E}[Y(k)]$$

if the limit exists. We note that for $q \neq r$, by using the ergodicity condition (3) and (6) we have

$$\mathsf{E}^*[a_{qr}^{\text{mn}}] = \sum_{i=1}^{|V^{\text{rw}}|} \pi_i \pi_i$$

while $a_{qq}^{\text{mn}}(k) = 0$. So

$$\mathsf{E}^*[A^{\text{mn}}] = \pi^{\mathrm{T}}\pi[I_N - ee^{\mathrm{T}}]$$

where $I_N$ is the $N \times N$ identity matrix. (This notation is adopted throughout.) We note that $\mathsf{E}^*[A^{\text{mn}}]$ is not a binary matrix consisting of entries 0 and 1, and therefore cannot be described as an adjacency matrix. However, it does provide a description of the time-averaged connectivity within the network. From (7), we have

$$\mathsf{E}^*[d_q^{\mathrm{mn}}] = \sum_{r=1, r\neq q}^{N} \sum_{i=1}^{|V^{\mathrm{rw}}|} \pi_i \pi_i$$

so that $\mathsf{E}^*[D^{\mathrm{mn}}] = (N-1)\pi^{\mathrm{T}}\pi I_N$.

Therefore, we associate our notion of long-time expected graph with a weighted Laplacian matrix given by

$$\mathsf{E}^*[L^{\mathrm{mn}}] = \pi^{\mathrm{T}}\pi[NI_N - e_N e_N^{\mathrm{T}}] \tag{8}$$

The weighted Laplacian (8) represents a weighted all-to-all coupling among the random walkers, see e.g. [13]. The eigenvalues of $\mathsf{E}^*[L^{\mathrm{mn}}]$ are 0 (with multiplicity 1) and $\pi^{\mathrm{T}}\pi N$ (with multiplicity $N-1$).

### B. Synchronization Problem

Let each agent carry an oscillator characterized by a $n$ dimensional autonomous dynamics. We study the time evolution of the complex dynamical system obtained by coupling the oscillators' dynamics according to the moving neighborhood network generated by the random walks. We assume that the random walkers do not move over time intervals of duration $\Delta > 0$. Jumps are allowed only at equally spaced transition instants $t_k = k\Delta, k \in \mathbb{Z}^+$. The resulting system dynamics is described by

$$\dot{x}_q(t) = f(x_q(t)) + \sigma B \sum_{r=1}^{N} l_{qr}^{\mathrm{mn}}(t) x_r,\ q = 1, \dots N,\ t \in \mathbb{R}^+ \tag{9}$$

where $t$ is the time variable, $x_q \in \mathbb{R}^n$ is the random state vector of the $q^{\mathrm{th}}$ agent, $f : \mathbb{R}^n \to \mathbb{R}^n$ describes the oscillators' individual dynamics, $B \in \mathbb{R}^{n\times n}$ describes coupling between oscillators, $\sigma$ is the coupling strength and $L^{\mathrm{mn}}(t)$ is the graph Laplacian of the moving neighborhood network. In the time interval $T_k = [t_k, t_{k+1})$ the random process $L^{\mathrm{mn}}(t)$ corresponds to the random variable $L^{\mathrm{mn}}(k)$. Thus it is a function of the random walks $X_1, \dots X_N$ at the $k^{th}$ time interval. We collect all the states of the system in the $nN$ dimensional vector $x$. The initial conditions are prescribed at the initial time $t = 0$ as $x(0) = x_0$.

If for any $t \in \mathbb{R}^+$ all the oscillators have the same state $s(t)$, that is

$$x_1(t) = ... = x_N(t) = s(t)$$

or equivalently

$$x = e_N \otimes s$$

we say that the agents are synchronized. Since $e$ is in the null space of $L^{\mathrm{mn}}$, a *synchronized solution* $s$ is a solution of the individual agent equation, namely $\dot{s} = f(s)$. The manifold in $\mathbb{R}^{nN}$ consisting of all trajectories $e_N \otimes s(t)$, where $s(t)$ is a solution of the individual agent equation is called the synchronization manifold.

Synchronization can be assessed by examining local stability of the oscillators with respect to the synchronization manifold. Linearizing each oscillator about the trajectory $s(t)$, which is assumed to be on the synchronization manifold yields,

$$\dot{z}_q(t) = F(t) z_q(t) + \sigma B \sum_{r=1}^{N} l_{qr}^{\mathrm{mn}}(t) z_j(t)$$

where

$$z_q(t) = x_q(t) - s(t)$$

and $F(t)$ is the Jacobian of $f$ evaluated at $s(t)$. Thus, the system of linearized coupled oscillators may be rewritten as

$$\dot{z}(t) = (I_N \otimes F(t) + \sigma L^{\mathrm{mn}}(t) \otimes B) z(t) \tag{10}$$

where $z(t) = [z_1^{\mathrm{T}}(t), \dots, z_N^{\mathrm{T}}(t)]^{\mathrm{T}}$. To assess asymptotic stability of the set of oscillators, we partition the state of (10) into a component that evolves along the synchronization manifold, and a component that evolves transverse to the synchronization manifold. For analysis, it suffices to show that the component that evolves transverse to the synchronization manifold asymptotically approaches the synchronization manifold.

Let $W \in \mathbb{R}^{N\times(N-1)}$ satisfy $W^{\mathrm{T}} e_N = 0$ and $W^{\mathrm{T}} W = I_{N-1}$. Note that the state vector $z(t)$ in (10) can be decomposed as $z(t) = (W \otimes I_n)\zeta(t) + e_N \otimes z_s(t)$, where $\zeta = (W^{\mathrm{T}} \otimes I_n) z$ is transverse to the synchronization manifold, and $z_s = \frac{1}{N}(e_N \otimes I_n)^{\mathrm{T}} z$ is on the synchronization manifold. Note that

$$((W \otimes I_n)\zeta)^{\mathrm{T}}(e_N \otimes z_s) = 0.$$

The variational equation (10) in terms of $\zeta$ and $z_s$ becomes

$$\dot{z}_s(t) = F(t) z_s(t) + \sigma(e_N^{\mathrm{T}} L^{\mathrm{mn}}(t) W \otimes B)\zeta(t)$$
$$\dot{\zeta}(t) = (I_{N-1} \otimes F(t) + \sigma W^{\mathrm{T}} L^{\mathrm{mn}}(t) W \otimes B)\zeta(t) \tag{11}$$

We say that the set of oscillators (9) locally asymptotically synchronize almost surely at the synchronized solution $s(t)$ if (11) is almost surely asymptotically stable, see e.g. [30], that is if $\zeta$ converges to zero almost surely for any $\zeta_0 \in \mathbb{R}^{(N-1)n}$ and for any initial probability $\pi_q$, q=1…N.. The definition of almost sure convergence may be found, for example, in Chapter 5 of [31]. System (11) represents a jump linear time varying system, see e.g. [32].

We associate to the stochastic dynamic network (9) the deterministic dynamic network

$$\dot{x}_q(t) = f(x_q(t)) + \sigma B \sum_{r=1}^{N} \mathsf{E}^*[l_{qr}^{\mathrm{mn}}] x_q(t),\ q = 1, \dots N,\ t \in \mathbb{R}^+ \tag{12}$$

where $\mathsf{E}^*[L^{\mathrm{mn}}]$ is the long-time expected value of the graph Laplacian defined in (8).

Synchronization of the deterministic set of coupled oscillators (12) may be studied using the master stability function. As a representative parameter for the synchronizability of (12), we introduce the *friendliness* $\Phi$ of the graph $G^{\mathrm{rw}}$ defined by $\Phi = \|\pi\|_2^2$ where $\|\cdot\|_2$, is the Euclidean norm. The stability question reduces by linear perturbation analysis to a constraint upon the coupling parameter $\sigma$, the friendliness $\pi$ and the number of agents $N$ of the form $\Phi N \in S$, where $S$ is the stability region and is an interval of $\mathbb{R}^+$. For many oscillator dynamical system (see e.g. [13]) the stability region is a bounded interval of the type $S = (\alpha_1, \alpha_2)$. The parameters $\alpha_1$, $\alpha_2$ are given by the master stability function, which is a property of the individual oscillator dynamic equation and of the coupling matrix $B$. Therefore, synchronization of (12) is generally expressed as a constraint on the control parameter $\sigma$, that is

$$\frac{\alpha_1}{N\Phi} < \sigma < \frac{\alpha_2}{N\Phi}$$

For large values of $\Phi$ (highly friendly networks) the set of oscillators (12) synchronizes for small values of the coupling parameter $\sigma$. While, large coupling is required for achieving synchronization in unfriendly networks. In addition, we note that for a prescribed graph $G^{\text{rw}}$ synchronization for small coupling may also be possible by increasing the number of agents $N$.

Our main contribution is to show that if the static network in (12) supports synchronization, also the stochastic network (9) does if the random walkers are sufficiently fast, or equivalently if the switching period $\Delta$ is sufficiently small.

## IV. Synchronization through fast-switching

In this section we show that asymptotic synchronization is achieved almost surely if the deterministic network (12) asymptotically synchronizes and the random walkers are moving sufficiently fast. By means of Proposition 1, the synchronization problem for the network of oscillators described by (9) reduces to the analysis of synchronization over a static network. Thus we reduce the problem to one which has been extensively studied in the literature and may be addressed by using the well-known master-stability function (MSF) analysis, see e.g. [13]. The proof of our result may be found in [33].

*Theorem 1:* Consider the deterministic dynamic system

$$\dot{y}(t) = (I_{N-1} \otimes F(t) + \sigma W^{\mathrm{T}} \mathsf{E}^*[L^{\text{mn}}] W \otimes B) y(t) \qquad (13)$$

representing the linearized transverse dynamics of (12). Assume that $F(t)$ is bounded and continuous in $\mathbb{R}^+$. If (13) is uniformly asymptotically stable, there is a time-scale $\Delta^* > 0$ such that for any shorter time-scale $\Delta < \Delta^*$ the stochastic system (9) locally asymptotically synchronizes almost surely.

## V. Illustration by numerical simulation

For the purpose of illustration, we consider a set of $N = 20$ agents diffusing in the small-world planar graph $G^{\text{rw}}$ with $|V^{\text{rw}}| = 50$ sites depicted in Fig. 1.

The graph $G^{\text{rw}}$ is connected and not-bipartite. The friendliness of the network is $\Phi = 0.020$.

Each agent is equipped with a Rössler oscillator. When agents occupy the same site, their first state is coupled. Thus

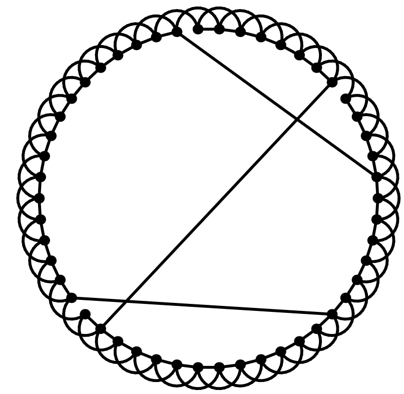

Fig. 1. Small-world graph $G^{\text{rw}}$ used for the numerical illustration.

the system of oscillators is described by

$$\dot{x}_{q1}(t) = -x_{q2}(t) - x_{q3}(t) - \sigma \sum_{r=1}^{N} l_{qr}^{\text{mn}} x_{j1}(t)$$
$$\dot{x}_{q2}(t) = x_{q1}(t) + a x_{q2}(t)$$
$$\dot{x}_{q3}(t) = b + x_{q3}(t)(x_{q1}(t) - c)$$

where $q = 1, \ldots, N$, and $a, b, c$ are constants.

By choosing the parameters $a = 0.2$, $b = 0.2$, and $c = 7$ from the stability region plot of Fig. 2 of [13] we have that the stability region $\alpha_1$ and $\alpha_2$ in (III-B) are $\alpha_1 = 0.2$ and $\alpha_1 = 2.3$. Therefore, from (III-B) the deterministic system (12) asymptotically synchronizes in the sense of the transverse Lyapunov exponents if $0.50 < \sigma < 5.7$. As previously remarked, this does not always mean the transverse dynamics is uniformly asymptotically stable. We choose $\sigma = 2$. Fig. 2 depicts the time evolution of the $x_1$ coordinate of the Rössler oscillators with static coupling given by the long-time expected graph.

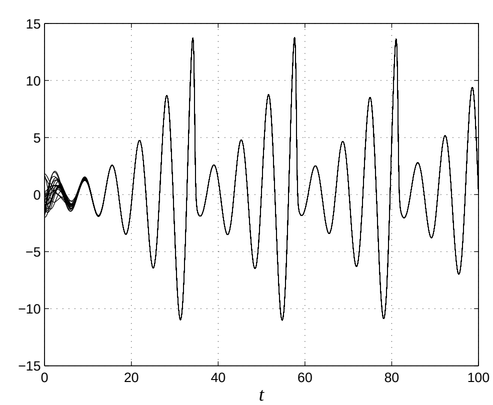


Fig. 2. Time evolution of the $x_1$ coordinate of the set of coupled Rössler oscillators using the long-time expected graph. Observe the asymptotic stability of the synchronized state.

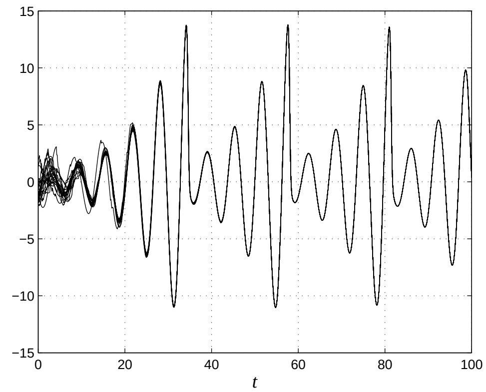


Fig. 3. Time evolution of the $x_1$ coordinate of the set of coupled Rössler oscillators using the moving neighborhood graph. Even though the neighborhood graph is instantaneously disconnected, fast switching allows for synchronization.

For the stochastic network (14) we consider a switching period for the random walkers of $\Delta = 0.1$. Fig. 3 depicts the $x_1$ coordinate of the set of coupled Rössler oscillators coupled by the moving neighborhood graph using the same initial conditions as in Fig. 2.

## VI. Conclusions

New generalizations on synchronization of mutually coupled oscillators are presented. We pose the synchronization problem in a stochastic dynamic framework where each agent diffuses in a finite lattice and carries an oscillator. The communication network topology evolves in time and is determined by the agents' locations in the lattice. Communication takes place only within geographical neighborhoods. We introduce the concept of long-time expected communication network defined as the ergodic limit of the stochastic time-varying network. We utilize tools based on fast switching and stochastic stability, and show that synchronization is asymptotically achieved if the long-time expected network supports synchronization and if the agents are moving sufficiently fast in the lattice. A numerical simulation illustrates the theoretical achievements of the present paper. We expect that theoretical framework presented in this paper to provide a better understanding of synchronization problems in biological, epidemiological and social networks, where the dynamics of the agent cannot be ignored.

## References


[1] J. M. Gonzalez-Miranda, *Synchronization and Control of Chaos*. London: Imperial College Press, 2004.

[2] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou, "The synchronization of chaotic systems," *Physics Reports*, vol. 366, pp. 1–101, 2002.

[3] D. He and L. Stone, "Spatio-temporal synchronization of recurrent epidemics," *Proceedings of the Royal Society of London. Series B*, vol. 270, pp. 1519–1526, 2003.

[4] E. Mosekilde, Y. Maistrenko, and D. Postnov, *Chaotic Synchronization: Applications to Living systems*. New Jersey: World Scientific, 2002.

[5] M. F. Laguna, G. Abramson, and D. H. Zanette, "Minorities in a model for opinion formation," *Complexity*, vol. 9, no. 4, pp. 31–36, 2004.

[6] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence*. Berlin: Springer, 1984.

[7] J. Ohtsubop, "Feedback induced instability and chaos in semiconductor lasers and their applications," *Optical Review*, vol. 6, no. 1, pp. 1–15, 1999.

[8] G. S. Duane, P. J. Webster, and J. B. Weiss, "Go-occurrence of northern and southern hemisphere blocks as partially synchronized chaos," *Journal of Atmospheric sciences*, vol. 56, no. 24, pp. 4183–4205, 1999.

[9] L. M. Pecora, T. L. Carroll, G. A. Johnson, and D. J. Mar, "Fundamentals of synchronization in chaotic systems, concepts, and applications," *Chaos*, vol. 7, no. 4, pp. 520–543, 1997.

[10] M. Barahona and L. M. Pecora, "Synchronization in small-world systems," *Physical Review Letters*, vol. 89, no. 5, p. 050101, 2002.

[11] M. Chen and D. Zhou, "Synchronization in uncertain complex network," *Chaos*, vol. 16, no. 1, p. in press, 2006.

[12] L. Kocarev and P. Amato, "Synchronization in power-law networks," *Chaos*, vol. 15, no. 2, p. 024101, 2005.

[13] L. Pecora, T. Carroll, G. Johnson, G. Mar, and K. S. Fink, "Synchronization stability in coupled oscillator arrays: Solution for arbitrary configurations," *International Journal of Bifurcation and Chaos*, vol. 10, p. 2, 2000.

[14] L. M. Pecora and T. L. Carroll, "Master stability functions for synchronized coupled systems," *Physical Review Letters*, vol. 80, no. 10, pp. 2109–2112, March 1998.

[15] C. W. Wu and L. O. Chua, "Synchronization in an array of linearly coupled dynamical systems," *IEEE Transactions on Circuits and Systems - I: Fundamental Theory and Applications*, vol. 42, no. 8, pp. 430–447, August 1995.

[16] C. Zhou, A. E. Motter, and J. Kurths, "Universality in the synchronization of weighted random networks," *Physical Review Letters*, vol. 96, no. 3, p. 034101, 2006.

[17] L. Kocarev, P. Janjic, U. Parlitz, and T. Stojanovski, "Controlling spatiotemporal chaos in coupled oscillatots by sporadic driving," *Chaos, Solitons & Fractals*, vol. 9, no. 1/2, pp. 283–293, 1998.

[18] T. Shibata and K. Kaneko, "Coupled mab gas: structures formation and dynamics of interacting motile elements with internal dynamics," *Physica D: Nonlinear Phenomena*, vol. 181, pp. 197–214, 2003.

[19] J. Lu, X. Yu, and G. Chen, "Chaos synchronization of general complex dynamical networks," *Physica A: Statistical Mechanics and its Applications*, vol. 334, pp. 281–302, 2004.

[20] J. Lu and G. Chen, "A time-varying complex dynamical network model and its controlled synchronization criteria," *IEEE Transactions on Automatic Control*, vol. 50, no. 6, pp. 841–846, 2005.

[21] I. V. Belykh, V. N. Belykh, and M. Hasler, "Blinking model and synchronization in small-world networks with a time-varying coupling," *Physica D: Nonlinear Phenomena*, vol. 195, no. 1-2, pp. 188–206, 2004.

[22] J. D. Skufca and E. M. Bollt, "Communication and synchronization in disconnected networks with dynamic topology: Moving neighborhood networks," *Mathematical Biosciences and Engineering*, vol. 1, no. 2, pp. 347–359, September 2004.

[23] D. J. Stilwell, E. M. Bollt, and D. G. Roberson, "Sufficient conditions for fast switching synchronization in time varying network topologies," *SIAM Journal on Applied Dynamical Systems*, Accepted for publication.

[24] P. Bremaud, *Markov Chains, Gibbs Fields, Monte Carlo Simulation, and Queues*, Springer-Verlag, Ed., 1999.

[25] P. Billingsley, *Probability and Measure*. New York: Wiley, 1986.

[26] R. Diestel, *Graph Theory*. New York, NY: Springer-Verlag, 1997.

[27] C. Godsil and G. Royle, *Algebraic Graph Theory*. New York, NY: Springer-Verlag, 2001.

[28] W. N. Anderson and T. D. Morley, "Eigenvalues of the laplacian of a graph," *Linear and Multilinear Algebra*, vol. 18, pp. 141–145, 1985.

[29] L. Lovasz, *Combinatorics, Paul Erdos is Eighty, Vol.2*. Janos Bolyai Mathematical Society, , Budapest, 1996, ch. Random walks on graphs: a survey, p. 353398.

[30] H. J. Kushner, *Stochastic Stability and Control*. New York: Academic Press, 1967.

[31] A. Gut, *Probability: A Graduate Course*. New York, NY: Springer, 2005.

[32] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov Jump Linear systems*. London, UK: Springer, 2005.

[33] M. Porfiri, D. J. Stilwell, E. M. Bollt, and J. D. Skufca, "Random talk: Random walk and synchronizability in a moving neighborhood network," *Physica D: Nonlinear phenomena*, to appear.