Edge of Chaos in a Parallel Shear Flow

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We study the transition between laminar and turbulent states in a Galerkin representation of a parallel shear flow, where a stable laminar flow and a transient turbulent flow state coexist. The regions of initial conditions where the lifetimes show strong fluctuations and a sensitive dependence on initial conditions are separated from the ones with a smooth variation of lifetimes by an object in phase space which we call the ''edge of chaos.'' We describe techniques to identify and follow the edge, and our results indicate that the edge is a surface. For low Reynolds numbers we find that the surface coincides with the stable manifold of a periodic orbit, whereas at higher Reynolds numbers it is the stable set of a higher-dimensional chaotic object.

In planar shear flows like plane Couette flow or pipe flow, turbulent dynamics may appear despite the linear stability of the laminar flow [1]. In these systems, sufficiently small perturbations to the laminar flow condition simply decay away, while slightly larger perturbations may result in turbulence. Experiments and numerics show that near onset, the turbulence is transient, sometimes persisting for a very long time and then suddenly decaying to the laminar profile [2–8]. The median lifetime of the transient increases rapidly with Reynolds number (Re) and may become longer than typical observation times, even at moderate Re. Both experimental and numerical evidence support an interpretation that the transients are due to a chaotic saddle [2,6]. Several low-dimensional models, based on Galerkin method, have been used to better understand this chaotic saddle [3,4,9–11]. Using the 9-variable model of [11], we were able to compute Lyapunov exponents and confirm a link between lifetimes and dimension of the chaotic saddle [12–14].

Previous work [2–5] suggests that transient behavior depends sensitively upon the initial condition, with a well-defined envelope to the chaos. As a gross descriptor, one can measure the duration of the transient for some initial state—the ''lifetime.'' Some regions of phase space are characterized by slowly varying lifetimes, with no sensitive dependence and no long transients. The chaotic saddle, indicated by rapid fluctuations in the lifetime function, appears to be confined in phase space by a geometric structure. We call this structure *the edge of chaos*, so named because chaotic trajectories come arbitrarily close. A plot of ''lifetime'' over a set of initial conditions (see Fig. 1) provides a visualization of the edge.

The purpose of this Letter is twofold: (1) to outline a new technique to calculate the edge, and (2) to present evidence that *the edge is a surface*, with interesting invariant structures embedded within. For Re < 402, the surface coincides with the stable manifold of a symmetric pair of periodic orbits. A similar phenomena has been identified in planar maps [such as the forced damped pendulum [15]], where the ''edge'' [16] between basins of attraction is formed by the stable manifold of a periodic orbit. To our knowledge, such structures have not previously been identified in higher-dimensional systems or systems with a single basin of attraction. Additionally, we find that as the Reynolds number is increased beyond Re = 402, although the edge structure continues to exist as a saddle surface in phase space, trajectories on the edge are no longer asymptotically periodic, but chaotic. The resultant limit set of these edge trajectories is a high-dimensional, fractal object embedded in the edge surface.

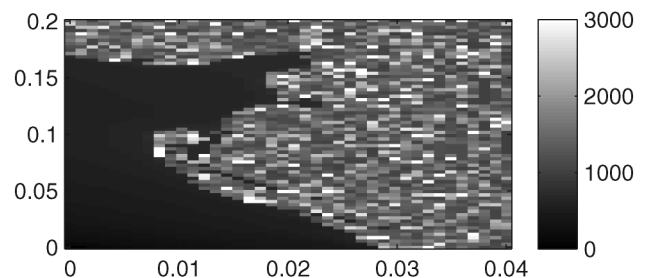Because characterization of the transition boundary requires extensive numerical simulations, we chose to de-

FIG. 1. *Edge of chaos.*—A 2-dimensional sample of phase space, with the laminar state at the origin. Each grid square is colored to show the transient lifetime for the trajectory with initial condition at the center of the square. The speckled region shows wildly varying transient behavior of the system and indicates that the sampled points are near the chaotic saddle. The smooth region appears to be clearly separated from the speckled, and we call that separating boundary *the edge*. Axis coordinates are magnitudes of the two randomly selected orthogonal vectors that generate the sample space.

velop our ideas using the 9D Galerkin projection of [11], which was derived from the 19-variable model studied in [3] by restricting the dynamics to an invariant symmetry subspace. The general structure of this class of n-dimensional models is that of ordinary differential equations with linear damping, quadratic coupling and a constant forcing:

$$\dot{x}_i = -\frac{d_i}{\mathrm{Re}}x_i + \sum_{j,k} a_{i;j,k}x_j x_k + f_i, \qquad i = 1., \ldots, n,$$

(1)

where d_i are the model dampings (essentially viscous), $a_{()}$ are the coefficients for the quadratic nonlinear coupling from the $u\nabla u$ term, and f_i stands for the forcing amplitude, which in our case is limited to one mode, driving a single parallel shear mode. Besides the Reynolds number, which controls the damping, there are two geometric parameters determining the widths and length of the flow domain. Following [3], we take the height of the cell to be 2, the width to be π, and the length to be 2π. The laminar profile is a fixed point of the system. By linear change of coordinates, we translate the system to place the attracting laminar state at the origin. We denote this new system $\dot{y} = Q(y; \mathrm{Re})$, indicating that the right-hand side is quadratic in y and studied over the parameter Re.

The lifetime of an initial condition, denoted $L(y_0)$, is defined as the time it takes the trajectory to come within a small distance ϵ of the laminar profile. By theorems on uniqueness of solutions to differential equations, each initial condition has a unique lifetime. Points of finite lifetime are in the *laminar basin*. A point whose trajectory never approaches the laminar profile has an infinite lifetime and is said to be in the *saddle set*.

In low-dimensional models [3,4] as well as fully resolved simulations [2,5], the lifetime function has a consistent character: As we increase distance from the laminar profile, the lifetime increases, first slowly and then very rapidly. Beyond a certain point (the *edge*), lifetime fluctuates wildly, with small intervals of smooth lifetimes interspersed (see Fig. 2). This behavior can be considered a typical "lifetime landscape" [3,4,17] for a chaotic

saddle. As illustrated in Fig. 3, the saddle set is bounded away from the attracting origin by the edge structure.

Sampling of the lifetime function is a standard approach and reveals some characteristic of both the saddle and the edge. From results of [12,14] on dimension of the stable set of the saddle, it is known that a typical line through phase space will intersect the saddle on a measure-0 Cantor set, with the lifetime diverging at each intersection. Edge points are associated with the end points of the "removed intervals" of the usual Cantor set construction. Each "removed interval" is a segment of points that lie in the laminar basin, yet any neighborhood of the ends of these intervals contains an uncountable infinity of points in the saddle set. For the range of Reynolds numbers considered, the Cantor set is much more dense than the middle-thirds construction, and the "small" [18] laminar intervals are difficult to resolve.

The primary weaknesses of sampling are: (1) sampling will not be sufficiently dense in a high-dimensional space; and (2) there are no dynamics associated with the sampled set. Whereas sampling looks at a fixed region of phase space, we can gain additional insight by considering the behavior of the edge under the flow of the differential equation. A simple continuity argument shows that the trajectory of an edge point must remain on the edge (an *edge trajectory*). By analyzing edge trajectories, we observe the dynamical structure that creates the edge.

Approximation of edge trajectories.—Because the edge trajectory is unstable, standard numerical integration cannot provide satisfactory approximations. Our approach provides a tractable solution to overcome this difficulty. We are confident that the technique has wider application to a broader class of problems. We outline our technique below.

Tracing a simple path from the origin to the chaotic saddle, there must be a *first* intersection of the edge. A point on the path *before* we reach the edge will have a trajectory whose amplitude remains small as it relaxes to the origin. However, a point on the path *after* we cross the edge will generate a chaotic transient, which typically contains at least one large amplitude excursion before
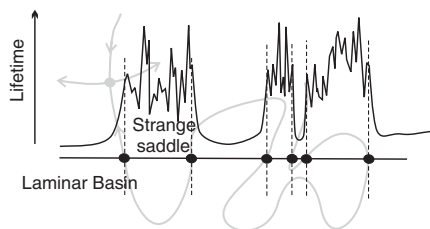

FIG. 2. *Lifetime indicates the edge.*—Lifetime $L(y)$ sampled along a line. In the laminar basin, $L(y)$ is smooth, while in the saddle region, it appears fractal. The boundary between those behaviors is an edge point. The gray curve illustrates that the behavior on the sampling line is related to larger structures in phase space.
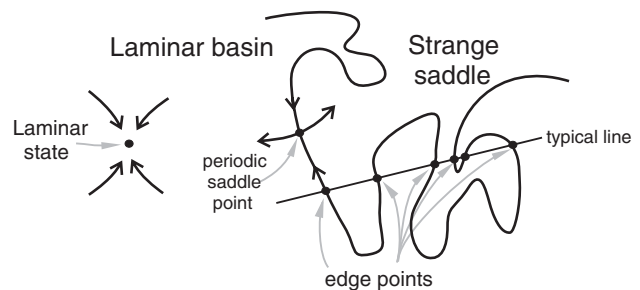

FIG. 3. *The edge of chaos.*—This cartoon schematically illustrates the *edge of chaos*, which separates the laminar basin from the strange saddle. The picture is representative of small Reynolds number, where we find that edge points lie on the 8-dimensional stable manifold of a periodic saddle orbit.
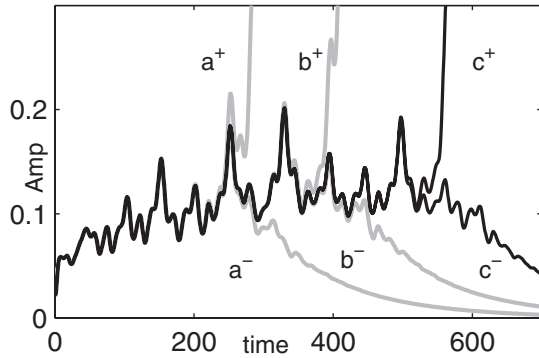
FIG. 4. *High-side and low side pairs.*—Trajectory amplitude as a function of time plotted for three pairs of nearby initial conditions. Trajectories labeled "−" are on the *low side*, and those with "+" on the *high side*. The initial conditions for the "a" pair were separated by $\approx 10^{-7}$. The pairs "b" and "c" result from refining the a pair (using bisection) to separations of $\approx 10^{-10}$ and $\approx 10^{-13}$, respectively. The limit of the bisection algorithm (in infinite precision) would yield a trajectory which would remain bounded away from the origin, but would never achieve a large amplitude typical of chaotic transients. The data shown are for Re = 390.

decaying. We classify an initial condition y as either on the *high side* or the *low side* based on whether the norm of its trajectory ($\max_t \| \phi(y;t) \|$) is above or below an appropriately chosen threshold value. To apply these ideas, we start with a low side point (near the origin) and a high side point (a chaotic transient). Any path that connects them must intersect the edge. By repeated bisection, we reduce the distance between the high-low pair to approximate the edge point that lies between them. This technique is much more efficient than trying to find the point of transition from smooth to fractal lifetimes and has proven very robust in numerical implementation. Figure 4 shows how bisection leads to increasingly accurate approximations of an edge trajectory. Because of the positive Lyapunov exponent associated with the unstable edge, a numerical initial condition will not generate the long trajectory which we require. As illustrated in Fig. 5, we apply techniques similar to the proper-interior-maximum-triple method [19] to generate arbitrarily long numerical approximations to the edge trajectory by successive refinement at suitable time intervals.

Structure from edge trajectories.—In small models such as the two or 4 coupled ordinary differential equations of [10,20], the boundary of the laminar basin is defined by the stable manifold of a fixed point that appears in a saddle node bifurcation, and the structure in phase space is relatively simple. In higher dimensions, these flow models typically have a rich bifurcation behavior, and boundary orbits (equivalent to the entire saddle set) are less clearly structured. However, the invariant subset of the saddle defined by the edge provides an identifiable structure which can be resolved by examining edge trajectories.

For Re $\lesssim$ 402, we find that a numerical edge trajectory converges to a periodic orbit, which we denote as p^+.
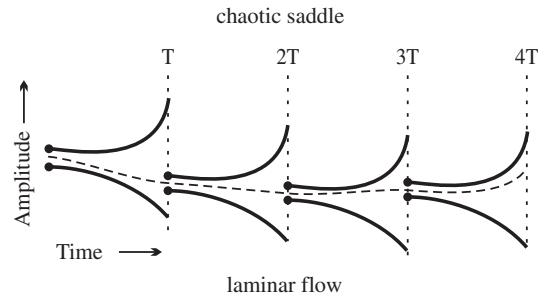

FIG. 5. *Numerical edge trajectory.*—At time 0, we start with two nearby initial conditions, one on each side of the edge. As the trajectories evolve, they are repelled from the edge, and we begin to lose precision in our approximation. At time T, before the error grows large, we use bisection to find a new pair of nearby initial conditions that are closer to the edge. By controlling refinement precision and interval T, we ensure the approximation maintains desired accuracy throughout the trajectory.

Because of a symmetry of the equations of motion, periodic orbits occur in pairs, and we denote the symmetry orbit as p^-. An example pair is shown in Fig. 6. These edge periodic orbits are unstable in only one direction, creating 8-dimensional stable manifolds $W^s_{p^+}$ and $W^s_{p^-}$, which are surfaces in 9-dimensional space. As the Reynolds number increases, the edge orbit undergoes period doubling and period halving bifurcations. At these bifurcations, the "old" periodic orbit becomes unstable in two directions, and the "new" edge orbit emerges with an 8-dimensional stable manifold. Simulations indicate that for each value of Reynolds number in this range, there is a unique periodic orbit pair with 8-dimensional stable manifold such that numerical edge trajectories converge to one or the other member of that pair, and therefore, the essential part of the edge is formed by the union of $W^s_{p^+}$ and $W^s_{p^-}$.

Above Re $\approx$ 402, edge trajectories no longer converge to a periodic orbit. At the bifurcation, the old edge orbit becomes unstable in two directions, but no new periodic orbit emerges with an 8-dimensional stable manifold. *Throughout* the parameter range considered, the edge set
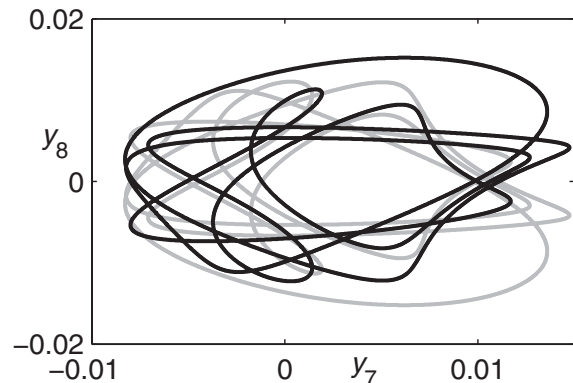

FIG. 6. *Periodic orbit pair for* Re = 390, graphed by plotting the y_7 and y_8 components over one period. Edge trajectories will asymptotically approach either the black or the gray orbit.
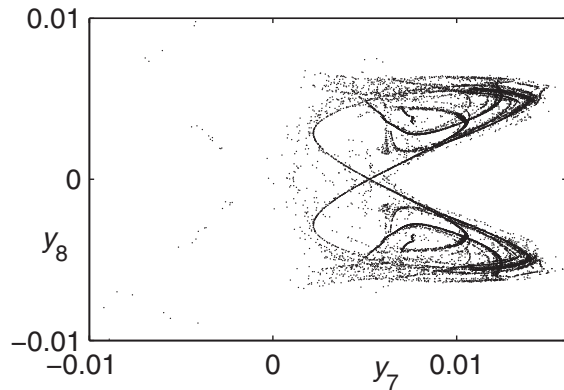
FIG. 7. *The relative attractor.*—From a long edge trajectory at Re = 420, we use the zero crossing of y_4 to construct a Poincare section. The graph shows only the y_7 vs y_8 components. The data appear to have the fractal structure characteristic of chaotic attractors.

appears to vary continuously in phase space with changes in the parameter, despite any bifurcations. The invariant saddle object that persists appears to be the union of surfaces that are smooth deformations of the manifolds $W^s_{p^+}$ and $W^s_{p^-}$ which existed before the bifurcation. In this parameter range, edge trajectories are numerically chaotic, with two positive Lyapunov exponents. We conjecture that the leading Lyapunov exponent is transverse to the edge, while the second positive exponent can be associated with the observed chaos on the edge. Edge trajectories converge to a more complicated invariant set which we call a *relative chaotic attractor* because it attracts nearby edge trajectories, while the edge itself remains unstable. Figure 7 illustrates that edge trajectories approach some higher-dimensional object instead of being asymptotically periodic.

Between the folds of the envelope.—Because the edge appears to be an 8-dimensional surface which could separate phase space, a reasonable question, then, is: "how do chaotic transients return to the origin?" In answer, we note the following: the edge has two symmetric parts intertwined in a complex fashion, repeatedly folded throughout phase space. Locally, we can treat this object as separating phase space. However, because of the fractal structure, when we "cross" the edge, we make an infinite number of crossings. Points on the saddle (edge) have infinite lifetime, but are a measure-0 set and are not detected in either experiment or simulation. By definition, a chaotic *transient* has finite lifetime; its initial condition lies *close* to the saddle but *in* the laminar basin. A point in the basin is contained in an open region that lies *between* the two symmetric parts of the edge.

Concluding remarks.—The edge of chaos described here is significant for issues such as control of turbulence, since it separates the laminar from the turbulent.

Transitions from the laminar to the turbulent state and vice versa will have to pass close to the edge of chaos described here. It is remarkable that even though the turbulent state and its almost space filling basin of attraction are high dimensional, the edge orbits seem to be of much lower dimension.

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