Mathematical Modelling and Analysis
Volume 28, Issue 1, 1–18, 2023
https://doi.org/10.3846/mma.20XXXX.xxxxxx

A Reproducing Kernel Method for Solving Singularly Perturbed Delay Parabolic Partial Differential Equations

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Received April 20, 2022; revised April 13, 2023; accepted 2023

Abstract. In this article, we put forward an efficient method on the foundation of a few reproducing kernel spaces(RK-spaces) and the collocation method to seek the solution of delay parabolic partial differential equations(PDEs) with singular perturbation. The approximated solution $\widetilde{g}_n(s,t)$ to the equations is formulated and proved the exact solution is uniformly convergent by the solution. Furthermore, the partial differentiation of the approximated solution is also proved the partial derivatives of the exact solution is uniformly convergent by the solution. Meanwhile, we show that the accuracy of our method is in the order of T/n where T is the final time and n is the number of spatial (and time) discretization in the domain of interests. Three numerical examples are put forward to demonstrate the effectiveness of our presented scheme.

Keywords: delay parabolic equation, reproducing kernel method, collocation method, numerical solution.

AMS Subject Classification: 35K20; 46E23; 65L60.

1 Introduction

The solutions of delay parabolic PDEs with singular perturbation at a limiting value of the singular parameter are different in character from the solutions of the general problem. This kind of PDEs are frequently used in varied forms of real-world applications, such as in the modeling of the human pupil-light reflex [22], population dynamics in mathematical biology, medicine and others [1, 29, 32, 40].

The PDEs with singular perturbation have been broadly studied by many scholars, including least squares method in [2], finite difference scheme in [4,26], Galerkin finite element method in [19], domain decomposition scheme in [20], reproducing kernel method(RKM) in [12] and others [6,27,30]. There are many references on numerical methods and numerical stability for delay differential equations, such as [5, 15, 17] to just list a few. Furthermore, finite difference schemes for PDEs with a time delay effect and a singular parameter are studied in 1D [3, 7, 14, 18] and in 2D [9] recently.

In this article, the following type of the singularly perturbed delay parabolic PDEs are considered by us

$$\begin{aligned} &\frac{\partial f(s,t)}{\partial t} - \varepsilon \frac{\partial^2 f(s,t)}{\partial s^2} + a(s,t)f(s,t) = F(s,t) - b(s,t)f(s,t-\tau), \quad (s,t) \in \Omega, \\ &f(0,t) = 0, \quad f(1,t) = 0, \quad t \in A_1, \\ &f(s,t) = \Psi(s,t), \quad (s,t) \in A_2, \end{aligned} \tag{1.1}$$

where $a(s,t) \geq 0$, $b(s,t) \geq \beta \geq 0$, $0 < \varepsilon \leq 1$, $\tau > 0$ and Ω, A_1, A_2 are $[0,1] \times [0,T]$, $[0,T]$, $[-\tau,0] \times [0,1]$, respectively. The forcing terms, $F(s,t)$ and $\Psi(s,t)$ are sufficiently smooth bounded functions, such that Equation (1.1) has a unique solution.

A robust finite difference method for the singularly perturbed delay parabolic PDEs are investigated by the authors in [3]. The focus of our paper, Equation (1.1) is a special case of model introduced in [3]. Thus, the theorems of uniqueness of the solutions to Equation (1.1) can be found in [3]. Additionally, we propose a RKM and collocation method to approximate the solutions to Equation (1.1) that does not require a separate time discretization scheme. Thus, it is more robust in terms of the discretization of temporal space. The RKM has attracted the interest of many authors. Xu and Lin [38] applied the RKM for solving the delay fractional differential equations. The RKM proposed by Geng and Cui [11] can be used to solve presented the RKM to solve the nonlocal fractional boundary value problems, in addition to the partial integro-differential equation, multi-point boundary value problems and so on, see [8,10,13,16,21,23,24,25,28,31,33,34,35,36,37,39,41] for more details. The aim of this article is to seek the approximate solutions of Equation (1.1) by the RKM and collocation method. Significantly, the Smith orthogonal process is averted and the computational time is saved by this method. Furthermore, the trouble cased by the delay term is dealt with in the established RK-space. Thus, it does not cost any computational expenses. Moreover, we can see that problem (1.1) has boundary layer behavior, it is important to obtain a proper approximation of the solutions for values where the boundary layer behavior is very severe. Therefore, we apply adaptive RKM to overcome this problem.

Structure of this thesis: a brief introduction is made with several applicable RK-spaces by us and its corresponding reproducing kernel function (RK-function) in Section 2. Section 3 presents a specific RKM and gives the approximated solution to Equation (1.1). Furthermore, astringency and error estimate of the numerical scheme are presented in Section 4. In Section 5, numerical examples are discussed to verify the effectiveness of the proposed method.

2 Preliminaries

In order to analyze the solution of Equation (1.1), we will present several RK-spaces in this section.

DEFINITION 1. Let $\mathbb{W}_1[0,1] = \{f(x) \mid f(x)$ be an absolutely continuous real-valued function in $[0,1]$, $f'(x) \in \mathbb{L}^2[0,1]\}$. In $\mathbb{W}_1[0,1]$, the $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$ are characterized by

$$\langle f,g\rangle_{\mathbb{W}_1} = f(0)g(0) + \int_0^1 f'(x)g'(x)dx,\ \forall\ f,\ g \in \mathbb{W}_1[0,1],$$
$$\|f\|_{\mathbb{W}_1} = \sqrt{\langle f,f\rangle_{\mathbb{W}_1}},\ \forall\ f \in \mathbb{W}_1[0,1],$$

respectively.

Lemma 1. *The functional space* $\mathbb{W}_1[0,1]$ *is a RK-space and its RK-function* $K_1(x,y)$ *has the following form*

$$K_1(x,y) = \begin{cases} x+1,\ x \le y, \\ y+1,\ x > y. \end{cases}$$

Proof. Similar to [8]. □

DEFINITION 2. Let $\mathbb{W}_2[0,T] = \{f(x) \mid f'(x)$ be an absolutely continuous real-valued function in $[0,T]$, $f''(x) \in \mathbb{L}^2[0,T]$, $f(0) = 0\}$. The $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$ are characterized by

$$\langle f,g\rangle_{\mathbb{W}_2} = f'(0)g'(0) + \int_0^T f''(x)g''(x)dx,\ \forall\ f,\ g \in \mathbb{W}_2[0,T],$$
$$\|f\|_{\mathbb{W}_2} = \sqrt{\langle f,f\rangle_{\mathbb{W}_2}},\ \forall\ f \in \mathbb{W}_2[0,T],$$

respectively.

Lemma 2. *The functional space* $\mathbb{W}_2[0,T]$ *is a RK-space and its RK-function* $K_2(x,y)$ *has the following form*

$$K_2(x,y) = \begin{cases} -\dfrac{1}{6}x^3 + \frac{1}{2}x^2y + xy,\ x \le y, \\ -\dfrac{1}{6}y^3 + \frac{1}{2}y^2x + xy,\ x > y. \end{cases}$$

Proof. Similar to [8]. □

DEFINITION 3. Let

$$\mathbb{W}_2^{'}[-\tau,T]=\{f(x)\mid -\tau\le t\le 0, u(0)=0, 0\le t\le T, f(x)\in\mathbb{W}_2[0,T]\}.$$

The $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are characterized by

$$\langle f,g\rangle_{\mathbb{W}_2'}=f_+^{'}(0)g_+^{'}(0)+\int_0^T f''(x)g''(x)dx,\ \forall\ f,\ g\in\mathbb{W}_2^{'}[-\tau,T],$$
$$\|f\|_{\mathbb{W}_2'}=\sqrt{\langle f,f\rangle_{\mathbb{W}_2'}},\ \forall\ f\in\mathbb{W}_2^{'}[-\tau,T],$$

respectively.

Lemma 3. *The space $\mathbb{W}_2^{'}[-\tau,T]$ is a RK-space and its RK-function $K_2'(x,y)$ has the following form*

$$K_2^{'}(x,y)=\begin{cases} K_2(x,y),\ 0\le x,y\le T,\\ 0,\ \textit{others}.\end{cases}$$

Proof. Similar to [8]. □

DEFINITION 4. Let $\mathbb{W}_3[0,1]=\{f(x)\mid f''(x)$ be an absolutely continuous real value function in $[0,1]$, $f'''(x)\in\mathbb{L}^2[0,1]$, $f(0)=f(1)=0\}$. The $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are characterized by

$$\langle f,g\rangle_{\mathbb{W}_3}=\sum_{i=1}^{2}f^i(0)v^i(0)+\int_0^1 f'''(x)g'''(x)dx,\ \forall\ f,\ g\in\mathbb{W}_3[0,1],$$
$$\|f\|_{\mathbb{W}_3}=\sqrt{\langle f,f\rangle_{\mathbb{W}_3}},\ \forall\ f\in\mathbb{W}_3[0,1],$$

respectively.

Lemma 4. *The functional space $\mathbb{W}_3[0,1]$ is a RK-space and its RK-function $K_3(x,y)$ has the following form*

$$K_3(x,y)=\begin{cases}-\dfrac{1}{18720}(x-1)y(156y^4+6x^2(y^4-5y^3+10y^2+30y+120)\\ -4x^3(y^4-5y^3+10y^2+30y+120)+x^4(y^4-5y^3+10y^2+30y+120)\\ +12x(3y^4-15y^3-100y^2-300y+360)),\qquad x\le y,\\ -\dfrac{1}{18720}(y-1)x(30xy(y^3-4y^2+6y-120)+10x^2y(y^3-4y^2\\ +6y-120)+120y(y^3-4y^2+6y+36)-5x^3y(y^3-4y^2\\ +6y+36)+x^4(y^4-4y^3+6y^2+36y+156)),\qquad x>y.\end{cases}$$

DEFINITION 5. Assume $\Omega=[0,1]\times[-\tau,T]$. Let $\mathbb{W}_{(3,2)}(\Omega)=\{f(s,t)\mid f_{sst}^{'''}$ be an absolutely continuous real-valued function in Ω, $f_{ssstt}^{(5)}\in\mathbb{L}^2(\Omega)$, $f(s,0)=f(0,t)=f(1,t)=0\}$. The $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are characterized by

$$\langle f,g\rangle_{\mathbb{W}_{(3,2)}}=\sum_{i=1}^{2}\int_0^T\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial s^i}f(0,t)\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial s^i}g(0,t)dt+\langle\frac{\partial}{\partial t}f(s,0),\frac{\partial}{\partial t}g(s,0)\rangle_{\mathbb{W}_3}$$
$$+\int_0^T\int_\Omega\frac{\partial^3}{\partial s^3}\frac{\partial^2}{\partial t^2}f(s,t)\frac{\partial^3}{\partial s^3}\frac{\partial^2}{\partial t^2}g(s,t)dsdt,\ \forall\ f,\ g\in\mathbb{W}_3[0,1]$$

and

$$\|f\|_{\mathbb{W}_3} = \sqrt{\langle f, f\rangle_{\mathbb{W}_3}},\ \forall\ f \in \mathbb{W}_3[0,1],$$

respectively.

Lemma 5. *The functional space $\mathbb{W}_{(3,2)}(\Omega)$ is a RK-space. Moreover, $\mathbb{W}_{(3,2)}(\Omega) = \mathbb{W}_3[0,1] \otimes \mathbb{W}_2^{'}[-\tau, T]$ and its RK-function $K_{(3,2)}(\bar{s}, \bar{t}, s, t)$ has the following form*

$$K_{(3,2)}(\bar{s}, \bar{t}, s, t) = K_3(\bar{s}, s)K_2^{'}(\bar{t}, t), \quad \forall\ (\bar{s}, s),\ (\bar{t}, t) \in \Omega.$$

Definition 6. Let $\Omega_1 = [0,1] \times [0,T]$. Let $\mathbb{W}_{(1,1)}(\Omega_1) = \{f(s,t) \mid f(s,t)$ be an absolutely continuous real-valued function in Ω_1, $f_{xt} \in \mathbb{L}^2[\Omega_1]\}$. Then, $\mathbb{W}_{(1,1)(\Omega_1)}$ is a RK-space and its RK-function $K_{(1,1)}(\bar{s}, \bar{t}, s, t)$ has the following form

$$K_{(1,1)}(\bar{s}, \bar{t}, s, t) = K_1(\bar{s}, s)K_1(\bar{t}, t), \quad \forall\ (\bar{s}, s),\ (\bar{t}, t) \in \Omega.$$

3 The RKM and collocation method for Equation (1.1)

The initial conditions of Equation (1.1) are brought into the RK-spaces, we must homogenize Equation (1.1). Let $g(s,t) = f(s,t) - \omega(s,t)$, where

$$\omega(s,t) = \begin{cases} \Phi(s,t), & -\tau \le t \le 0, \\ \Phi(s,0), & 0 \le t \le T. \end{cases}$$

Then, we can acquire a homogeneous system from Equation (1.1) as follows

$$\begin{cases} g(s,t) = 0, \quad \tau \le t \le 0, \\ \dfrac{\partial g}{\partial t} - \varepsilon \frac{\partial^2 g}{\partial s^2} + ag + bg(s, t-\tau) = F_1(s,t), \ \ 0 \le x \le 1,\ 0 \le t \le T, \\ g(0,t) = 0, \quad g(1,t) = 0, \quad 0 \le t \le T, \end{cases} \tag{3.1}$$

where

$$F_1(s,t) = \begin{cases} \varepsilon \dfrac{\partial^2}{\partial s^2}\Phi(s,0) - a(s,t)\Phi(s,0) - b(s,t)\Phi(s, t-\tau) + F(s,t),\ 0 \le t \le \tau, \\ \varepsilon \dfrac{\partial^2}{\partial s^2}\Phi(s,0) - a(s,t)\Phi(s,0) - b(s,t)\Phi(s,0) + F(s,t),\ \ t > \tau. \end{cases}$$

Let $\mathcal{B} : \mathbb{W}_{(3,2)}(\Omega) \to \mathbb{W}_{(1,1)}(\Omega_1)$ be a differential operator such that

$$\mathcal{B}g = \frac{\partial g}{\partial t} - \varepsilon \frac{\partial^2 g}{\partial s^2} + ag + bg(s, t-\tau), \quad \text{for } g(s,t) \in \mathbb{W}_{(3,2)}(\Omega).$$

Then, Equation (3.1) can be converted into the following form

$$\begin{cases} g(s,t) = 0, & -\tau \le t \le 0, \\ \mathcal{B}g(s,t) = F_1(s,t),\ 0 \le x \le 1, & 0 \le t \le T, \\ g(0,t) = g(1,t) = 0, & 0 \le t \le T. \end{cases} \tag{3.2}$$

The operator $\mathcal{B}$ will be proved which is linear differential operator with boundedness in the remainder of this section. Then we will form a basis for the RK-space $\mathbb{W}_{(3,2)}(\Omega)$ fabricated in the previous section. Therefore, we will approximate the solution of Equation (3.2) by a function sequence in $\mathbb{W}_{(3,2)}(\Omega)$.

Lemma 6. $\mathcal{B} : \mathbb{W}_{(3,2)}(\Omega) \to \mathbb{W}_{(1,1)}(\Omega_1)$ *is a bounded linear operator.*

Proof. It is obvious that $\mathcal{B}$ is a linear operator. We can obtain the boundedness if the following relation holds that

$$\|\mathcal{B}g(s,t)\|^2_{\mathbb{W}_{(1,1)}} \le M\|g\|^2_{\mathbb{W}(3,2)}, \quad M > 0.$$

Utilization of the reproducing property of RK-function $K_{(3,2)}(\bar{s}, \bar{t}, s, t)$, we can get

$$\begin{aligned}
&g(s,t) = \langle g(\cdot,\cdot), K_{(3,2)}(s,t,\cdot,\cdot)\rangle_{(3,2)},\\
&\partial^i_{s^i}\partial^j_{t^j}\mathcal{B}g(s,t) = \langle g(\cdot,\cdot), \partial^i_{s^i}\partial^j_{t^j}\mathcal{B}K_{(3,2)}(s,t,\cdot,\cdot)\rangle_{(3,2)}, \ \ i,j = 0,1.
\end{aligned}$$

Hence, we utilize $\partial^i_{s^i}\partial^j_{t^j}\mathcal{B}g(s,t)$ and the continuity of $K_{(3,2)}(s,t,\cdot,\cdot)$ as well as the Schwarz inequality, one can be written

$$\begin{aligned}
\partial^i_{s^i}\partial^j_{t^j}\mathcal{B}g(s,t) \mid &= \langle g(\cdot,\cdot), \partial^i_{s^i}\partial^j_{t^j}\mathcal{B}K_{(3,2)}(s,t,\cdot,\cdot)\rangle_{\mathbb{W}_{(3,2)}} \mid\\
&\le \|g\|_{\mathbb{W}_{(3,2)}}\|\partial^i_{s^i}\partial^j_{t^j}\mathcal{B}K_{(3,2)}(s,t,\cdot,\cdot)\|_{\mathbb{W}_{(3,2)}} \le M_{i,j}\|g\|_{\mathbb{W}_{(3,2)}}.
\end{aligned}$$

Make use of the inner product and the norm of $\mathbb{W}_{(3,2)}(\Omega)$, we can get that

$$\begin{aligned}
&\|\mathcal{B}g(s,t)\|^2_{\mathbb{W}_{(1,1)}} = \langle \mathcal{B}g(s,t), \mathcal{B}g(s,t)\rangle_{\mathbb{W}_{(1,1)}} = \int_0^T \left(\frac{\partial}{\partial t}\mathcal{B}g(0,t)\right)^2 dt\\
&+ \langle \mathcal{B}g(s,0), \mathcal{B}g(0,t)\rangle_{\mathbb{W}_1} + \iint_{\Omega_1}\left(\frac{\partial}{\partial s}\frac{\partial}{\partial t}\mathcal{B}g(s,t)\right)^2 dsdt = \int_0^T \left(\frac{\partial}{\partial t}\mathcal{B}g(0,t)\right)^2 dt\\
&+ (\mathcal{B}g(0,0))^2 + \int_0^1 \left(\frac{\partial}{\partial s}\mathcal{B}g(s,0)\right)^2 ds + \iint_{\Omega_1}(\frac{\partial}{\partial s}\frac{\partial}{\partial t}\mathcal{B}g(s,t))^2 dsdt\\
&\le \int_0^T M_0^2\|g\|^2_{\mathbb{W}_{(3,2)}}dt + M_1^2\|g\|^2_{\mathbb{W}_{(3,2)}} + \int_0^1 M_2^2\|g\|^2_{\mathbb{W}_{(3,2)}}ds + \iint_{\Omega} M_3^2\|g\|^2_{\mathbb{W}_{(3,2)}}dsdt\\
&\quad = (M_0^2 + M_1^2 + M_2^2T + M_3^2T)\|g\|^2_{\mathbb{W}_{(3,2)}}.
\end{aligned}$$

That is,

$$\|\mathcal{B}g(s,t)\|^2_{\mathbb{W}_1} \le M\|g\|^2_{\mathbb{W}_{(3,2)}},$$

where $M = M_0^2 + M_1^2 + M_2^2T + M_3^2T$. Thus, the linear operator $\mathcal{B}$ is bounded as well. □

Lemma 7. *Let*

$$\Phi_i(s,t) = K_{(1,1)}(s_i,t_i,s,t), \quad \Psi_i(s,t) = \mathcal{B}^*\Phi_i(s,t),$$

as suppose that $\{(s_i,t_i)\}_{i=1}^{\infty}$ *is dense on* Ω, *where* $\mathcal{B}^*$ *is the conjugate operator of* $\mathcal{B}$ *and* $K_{(1,1)}$ *is the RK-function of* $\mathbb{W}_{(1,1)}(\Omega_1)$. *Then,*

$$\Psi_i(s,t) = \mathcal{B}K_{(3,2)}(s_i,t_i,s,t).$$

Proof. Owing to the properties of the RK-function, we can get that

$$\begin{aligned}\Psi_i(s,t) &= \langle \mathcal{B}^* K_{(1,1)}(s_i,t_i,\cdot,\cdot), K_{(3,2)}(s,t,\cdot,\cdot)\rangle_{\mathbb{W}_{(3,2)}} \\ &= \langle K_{(1,1)}(s_i,t_i,\cdot,\cdot), \mathcal{B}K_{(3,2)}(s,t,\cdot,\cdot)\rangle_{\mathbb{W}_{(1,1)}} = \mathcal{B}K_{(3,2)}(s_i,t_i,s,t).\end{aligned}$$

This concludes the Lemma. □

Remark 1. By the Lemma above, we can get that

$$\begin{aligned}\Psi_i(s,t) \;\; = \;\; & \frac{\partial K_2'(t_i,t)}{\partial t_i} K_3(s_i,s) - \varepsilon \frac{\partial^2 K_3(s_i,s)}{\partial {s_i}^2} K_2'(t_i,t) \\ & + a(s,t)K_3(s_i,s)K_2'(t_i,t) + b(s,t)K_3(s_i,s)K_2'(t_i,t-\tau).\end{aligned}$$

Notice that the RK-functions K_2' and K_3 are symmetric, it follows that

$$\begin{aligned}&\langle \Psi_i(s,t), \Psi_j(s,t)\rangle = (\mathcal{B}\Psi_i(s,t))(s_j,t_j) \\ &\quad = \frac{\partial \Psi_i(s,t)}{\partial t_j} - \varepsilon \frac{\partial^2 \Psi_i(s,t)}{\partial {x_j}^2} + a(s,t)\Psi_i(s,t) + b(s,t)\Psi_i(s,t-\tau).\end{aligned}$$

Now we are ready to define a basis for the RK-space $\mathbb{W}_{(3,2)(\Omega)}$.

Theorem 1. *The sequence* $\{\Psi_i(s,t)\}_{i=1}^{\infty}$ *is linearly independent in* $\mathbb{W}_{(3,2)}(\Omega)$ *as suppose that* $\{(s_i,t_i)\}_{i=1}^{\infty}$ *is dense on* Ω*.*

Proof. If we can obtain that $\{\Psi_i(s,t)\}_{i=1}^{m}$ is linearly independent for any $m \geq 1$, this conclusion is obvious. Actually, if $\{c_i\}_{i=1}^{m}$ satisfies that

$$\sum_{i=1}^{m} c_i \Psi_i(s,t) = 0,$$

taking $\alpha_k(s,t)$ such that

$$\alpha_k(x_l,t_l) = \begin{cases} 1, & l = k, \\ 0, & l \neq k, \end{cases}$$

where $\alpha_k(s,t) \in \mathbb{W}_{(3,2)}(\Omega)$, for each $l = 1,2,\ldots,m$, then we can obtain that

$$\begin{aligned}0 \;\; &= \;\; \langle \alpha_k(s,t), \sum_{i=1}^{m} c_i \Psi_i(s,t)\rangle_{\mathbb{W}_{(3,2)}} = \sum_{i=1}^{m} c_i \langle \alpha_k(s,t), \Psi_i(s,t)\rangle_{\mathbb{W}_{(3,2)}} \\ &= \;\; \sum_{i=1}^{m} c_i \alpha_k(s_i,t_i) = c_k, \;\; k = 1,2,\ldots,m.\end{aligned}$$

Hence, we can arrive at a conclusion that $\{\Psi_i(s,t)\}_{i=1}^{m}$ is linearly independent for all $m \geq 0$. Therefore, $\{\Psi_i(s,t)\}_{i=1}^{\infty}$ is linearly independent in $\mathbb{W}_{(3,2)}(\Omega)$. □

The main theorem in this paper is given below. This theorem provides an approximated solution to Equation (3.2) in the RK-space $\mathbb{W}_{(3,2)}(\Omega)$.

Theorem 2. *Let* $S_n = span\{\Psi_1(s,t), \Psi_2(s,t), \cdots, \Psi_n(s,t)\}$ *and* $P_n : \mathbb{W}_{(3,2)}(\Omega) \to S_n$ *be the orthogonal projection operator of* $\mathbb{W}_{(3,2)}(\Omega)$ *onto* S_n*. If* $g(s,t)$ *is the solution of Equation* (3.2)*, then,* $\widetilde{g}_n(s,t) = P_n g$ *satisfies*

$$\langle \widetilde{g}_n, \Psi_i \rangle = F_1(s_i, t_i),\ i = 1, 2, \ldots, n. \tag{3.3}$$

Furthermore,

$$\widetilde{g}_n(s,t) = \sum_{j=1}^{n} a_j \Psi_j(s,t) \tag{3.4}$$

is an approximate solution, where $a_1, a_2, \ldots, a_n$ *are undetermined constants, which can be determined by*

$$\begin{pmatrix} \langle \Psi_1, \Psi_1 \rangle & \langle \Psi_2, \Psi_1 \rangle & \cdots & \langle \Psi_n, \Psi_1 \rangle \\ \langle \Psi_1, \Psi_2 \rangle & \langle \Psi_2, \Psi_2 \rangle & \cdots & \langle \Psi_n, \Psi_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle \Psi_1, \Psi_n \rangle & \langle \Psi_2, \Psi_n \rangle & \cdots & \langle \Psi_n, \Psi_n \rangle \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} F_1(s_1, t_1) \\ F_1(s_2, t_2) \\ \vdots \\ F_1(s_n, t_n) \end{pmatrix}.$$

Proof. Owing to the properties of the RK-function and the self-conjugation of the operator P_n, it can be shown that

$$\begin{aligned} \langle P_n g, \Psi_i \rangle &= \langle g, P_n \Psi_i \rangle \ \Psi \text{ self-conjugate} \\ &= \langle g, \Psi_i \rangle \ \Psi \text{ orthogonal projection} \\ &= \langle g, \mathcal{B}^* \Phi_i \rangle \text{ Definition of } \Psi_i \\ &= \langle \mathcal{B} g, \Phi_i \rangle = \mathcal{B} g(s_i, t_i) = F_1(s_i, t_i). \end{aligned}$$

To gain the approximated solution $\widetilde{g}_n$ in the form of Equation (3.4), we substitute Equation (3.4) into Equation (3.3). Through collocation process, we have that

$$\sum_{j=1}^{n} a_j \langle \Psi_j(s,t), \Psi_i(s,t) \rangle = F_1(s,t),\ \forall\ i = 1, \ldots, n. \tag{3.5}$$

Rewrite the above system in a matrix form, we have that

$$\mathbf{Ga} = \mathbf{F_1}, \tag{3.6}$$

where

$$\mathbf{G} = \begin{pmatrix} \langle \Psi_1, \Psi_1 \rangle & \langle \Psi_2, \Psi_1 \rangle & \cdots & \langle \Psi_n, \Psi_1 \rangle \\ \langle \Psi_1, \Psi_2 \rangle & \langle \Psi_2, \Psi_2 \rangle & \cdots & \langle \Psi_n, \Psi_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle \Psi_1, \Psi_n \rangle & \langle \Psi_2, \Psi_n \rangle & \cdots & \langle \Psi_n, \Psi_n \rangle \end{pmatrix},$$

$$\mathbf{a} = \left(a_1 a_2 \cdots a_n\right)^T, \ \ \mathbf{F_1} = \left(F_1(s_1, t_1) F_1(s_2, t_2) \cdots F_1(s_n, t_n)\right)^T.$$

Then, we have that $\mathbf{a} = \mathbf{G}^{-1}\mathbf{F_1}$ as required. □

Algorithm:

Step 1. Calculating the RK-functions $K_{(1,1)}(\bar{s},\bar{t},s,t)$ and $K_{(3,2)}(\bar{s},\bar{t},s,t)$;

Step 2. Structuring a bounded linear operator $\mathcal{B}$;

Step 3. Structuring Ψ_i and the projection operator P_n;

Step 4. Setting up Equation (3.5) in the light of the projection operator, and expressed as matrix form;

Step 5. Finding the corresponding coefficients in Equation (3.6).

Consider the domain $\Omega = [0,1]\times[0,T]$. Instead of using fixed collocation points on the domain Ω, we realize that an adaptive collocation points cross domain during the layer are critical to certain situations. We observe that there is a connection between the points that had a larger error of f_n and the points that had larger errors of F. This motivates us to use the error of F as an indicator for adding points.

In practice, we first select a set A of n points uniformly across the domain. By applying our proposed RKM to obtain an approximating solution. We then choose a different set $\mathcal{B}$ of $2n$ points randomly as test points. We calculate $\mathcal{B}f_n - F$ at the above $2n$ points of $\mathcal{B}$ and pick n points that give the highest error in predicting F. We add this set of points to previous collocation points and using the RKM again to obtain an approximation f_{2n}. This procedure is important, as it not only prevents us from losing the accuracy of the solution across the entire domain but also helps us to focus more points on the boundary layer.

4 Convergence and error estimation

Theorem 3. *As defined in Equation* (3.4)*,* $g(s,t)$ *is uniformly convergent by* $\widetilde{g}_n(s,t)$*.*

Proof. Obviously, $\|\widetilde{g}_n - g\| \to 0$ holds as $n\to\infty$. Like that, $\widetilde{g}_n(x)$ is the approximate solution of Equation (3.2). By the following inequalities

$$\|\widetilde{g}_n(s,t)-g(s,t)\| = \|\langle \widetilde{g}_n - g, K_{(3,2)}\rangle\| \le \|\widetilde{g}_n - g\|\|K_{(3,2)}\|,\ \ \|K_{(3,2)}\| \le M$$

since $K_{(3,2)}$ is continuous on $[0,1]$, where M is a real number and $M>0$, we can draw a conclusion that $g(s,t)$ is uniformly convergent by $\widetilde{g}_n(s,t)$ on $[0,1]$. □

Theorem 4. *The partial derivatives of the exact solution* $\partial^i_{t^i}\partial^j_{s^j}g(s,t)$ *are uniformly convergent by* $\partial^i_{t^i}\partial^j_{s^j}\widetilde{g}_n(s,t)$*, whenever* $i=0,1$ *and* $j=0,1,2$*, where* $\partial^i_{t^i}\partial^j_{s^j}\widetilde{g}_n(s,t)$ *are the partial derivatives of the numerical solution* $\widetilde{g}_n(s,t)$*.*

Proof. Since $\mathbb{W}_{(3,2)}$ is a Hilbert space, obviously, $\|\widetilde{g}_n - g\| \to 0$ holds as $n \to \infty$. Again, since

$$\begin{aligned}&\|\partial^i_{t^i}\partial^j_{s^j} g(s,t) - \partial^i_{t^i}\partial^j_{s^j}\widetilde{g}_n(s,t)\| \\ &\quad = \|\langle g(y,s) - \widetilde{g}_n(y,s), \partial^i_{t^i}\partial^j_{s^j}\mathcal{B}K_{(3,2)}(s,t,y,s)\rangle\|_{\mathbb{W}_{(3,2)}} \\ &\quad \le \|g - \widetilde{g}_n\|_{\mathbb{W}_{(3,2)}} \|\partial^i_{t^i}\partial^j_{s^j}\mathcal{B}K_{(3,2)}(s,t,y,s)\|_{\mathbb{W}_{(3,2)}} \le M_{i,j}\|g - \widetilde{g}_n\|_{\mathbb{W}_{(3,2)}},\end{aligned}$$

hence $\partial^i_{t^i}\partial^j_{s^j}\widetilde{g}_n(s,t)$ converges uniformly to $\partial^i_{t^i}\partial^j_{s^j} g(s,t)$. □

Next, we will give an error analysis on the approximated solution $\widetilde{g}_n$ to the true solution g for Equation (3.2).

Theorem 5. *Let a dense subset of the domain Ω be $S = \{(s_1,t_1),(s_2,t_2),\ldots\}$. Then,*

$$\mathcal{B}g(s_j,t_j) = \mathcal{B}\widetilde{g}_n(s_j,t_j), \quad (s_j,t_j) \in S, j \le n.$$

Proof. Owing to the properties of the RK-function and the self-conjugation of the operator P_n, we can get that

$$\begin{aligned}&\mathcal{B}\widetilde{g}_n(s_j,t_j) = \langle \widetilde{g}_n(\cdot,\cdot), \mathcal{B}K_{(3,2)}(s_j,t_j,\cdot,\cdot)\rangle \\ &\quad = \langle \widetilde{g}_n(\cdot,\cdot), \Psi_j(\cdot,\cdot)\rangle \text{ Definition of } \Psi \\ &\quad = \langle P_n g(\cdot,\cdot), \Psi_j(\cdot,\cdot)\rangle \; P_n \text{ self-conjugation} \\ &\quad = \langle g(\cdot,\cdot), P_n\Psi_j(\cdot,\cdot)\rangle = \langle g(\cdot,\cdot), \Psi_j(\cdot,\cdot)\rangle \\ &\quad = \langle g(\cdot,\cdot), \mathcal{B}K_{(3,2)}(s_j,t_j,\cdot,\cdot)\rangle = \mathcal{B}\langle g(\cdot,\cdot), K_{(3,2)}(s_j,t_j,\cdot,\cdot)\rangle = \mathcal{B}g(s_j,t_j).\end{aligned}$$

□

The error estimation of the approximated solution, through the following theorem, constructed by our RK-space $\mathbb{W}_{(3,2)}(\Omega)$, $\widetilde{g}_n$.

Theorem 6. *Recall T is the final time of interests, n is the sum of points in the domain Ω. Then,*

$$\|g(s,t) - \widetilde{g}_n(s,t)\| = \mathcal{O}\left(T/n\right).$$

Proof. For $\forall\ n \in N$ and $(s,t) \in \Omega$, take $(s_j,t_j) \in S$, $j \le n$, where $S = \{(s_1,t_1),(s_2,t_2),\ldots\}$, such that $\mid s - s_j \mid \le 1/n$ and $\mid t - t_j \mid \le T/n$. By Equation (5), we can arrive at

$$\begin{aligned}&\mathcal{B}\widetilde{g}_n(s,t) - \mathcal{B}g(s,t) = \mathcal{B}\widetilde{g}_n(s,t) - \mathcal{B}\widetilde{g}_n(s_j,t_j) - (\mathcal{B}g(s,t) - \mathcal{B}\widetilde{g}_n(s_j,t_j)) \\ &\quad = \langle \widetilde{g}_n(\cdot,\cdot), \mathcal{B}K_{(3,2)}(s,t,\cdot,\cdot) - \mathcal{B}K_{(3,2)}(s_j,t_j,\cdot,\cdot)\rangle \\ &\quad\ - \langle g(\cdot,\cdot), \mathcal{B}K_{(3,2)}(s,t,\cdot,\cdot) - \mathcal{B}K_{(3,2)}(s_j,t_j,\cdot,\cdot)\rangle \\ &\quad = \langle \widetilde{g}_n(\cdot,\cdot) - g(\cdot,\cdot), \mathcal{B}K_{(3,2)}(s,t,\cdot,\cdot) - \mathcal{B}K_{(3,2)}(s_j,t_j,\cdot,\cdot)\rangle.\end{aligned}$$

Furthermore, based on the reversible property of the operator $\mathcal{B}$, we have that

$$\begin{aligned}&\widetilde{g}_n(s,t) - g(s,t) = \langle \widetilde{g}_n - v, \mathcal{B}^{-1}(\mathcal{B}K_{(3,2)}(s,t,\cdot,\cdot) - \mathcal{B}K_{(3,2)}(s_j,t_j,\cdot,\cdot))\rangle \\ &\quad \le \|\mathcal{B}^{-1}\|\|\widetilde{g}_n(s,t) - g(s,t)\|\|\mathcal{B}K_{(3,2)}(s,t,\cdot,\cdot) - \mathcal{B}K_{(3,2)}(s_j,t_j,\cdot,\cdot)\|.\end{aligned}$$

From the defintion of $K_{(3,2)}(s,t,\bar{s},\bar{t})$, it can be seen that $\mathcal{B}K_{(3,2)}(s,t,\cdot,\cdot)$ is differentiable with respect to (s,t). Utilizing the mean value theorem with regard to s and t, respectively, we can get that

$$\begin{aligned}&\mathcal{B}K_{(3,2)}(s_i,t_i,s,t)-\mathcal{B}K_{(3,2)}(s_j,t_j,\cdot,\cdot)\\&\quad=\frac{\partial}{\partial\xi}\mathcal{B}K_{\xi,\eta}(\cdot,\cdot)(s-s_j)+\frac{\partial}{\partial\eta}\mathcal{B}K_{\xi,\eta}(\cdot,\cdot)(t-t_j).\end{aligned}$$

Thus,

$$\begin{aligned}&\widetilde{g}_n(s,t)-g(s,t)\le\|\mathcal{B}^{-1}\|\|\widetilde{g}_n(s,t)-g(s,t)\|s-s_j\|\frac{\partial}{\partial\xi}\mathcal{B}K_{\xi,\eta}(\cdot,\cdot)\|+\|\mathcal{B}^{-1}\|\|\widetilde{g}_n(s,t)\\&-g(s,t)\|t-t_j\|\frac{\partial}{\partial\eta}\mathcal{B}K_{\xi,\eta}(\cdot,\cdot)\|\le\frac{1}{n}\|\mathcal{B}^{-1}\|\|\widetilde{g}_n(s,t)-g(s,t)\|\|\frac{\partial}{\partial\xi}\mathcal{B}K_{\xi,\eta}(\cdot,\cdot)\|\\&+\frac{T}{n}\|\mathcal{B}^{-1}\|\|\widetilde{g}_n(s,t)-g(s,t)\|\|\frac{\partial}{\partial\eta}\mathcal{B}K_{\xi,\eta}(\cdot,\cdot)\|.\end{aligned}$$

Since both $\|\frac{\partial}{\partial\xi}\mathcal{B}K_{\xi,\eta}(\cdot,\cdot)\|$ and $\|\frac{\partial}{\partial\eta}\mathcal{B}K_{\xi,\eta}(\cdot,\cdot)\|$ are bounded, and $\|\widetilde{g}_n(s,t)-g(s,t)\|\to 0$, we conclude that

$$g(s,t)-\widetilde{g}_n(s,t)=\mathcal{O}\left(T/n\right).$$

□

5 Numerical results

In this section, we present some numerical experiments to verify our theoretical findings. We operate our programs in MATHEMATICA 13.0. In all examples, we first use a uniform meshes of n points on Ω. We compute the error $e_n = f_n - f$ in different type norms. For convenience, we denote

$$\begin{aligned}&\|e_n\|_0^2:=\int_\Omega(f(s,t)-f_n(s,t))^2\,\mathrm{d}s\mathrm{d}t,\ \|e_n\|_{1,t}^2:=\int_\Omega(\partial_t f(s,t)-\partial_t f_n(s,t))^2\,\mathrm{d}s\mathrm{d}t,\\&\|e_n\|_{1,s}^2:=\int_\Omega\left(\partial_s f(s,t)-\partial_s f_n(s,t)\right)^2\mathrm{d}s\mathrm{d}t,\\&\|e_n\|_{2,s}^2:=\int_\Omega\left(\partial_{ss} f(s,t)-\partial_{ss} f_n(s,t)\right)^2\mathrm{d}s\mathrm{d}t.\end{aligned}$$

Example 1. Let us examine the singularly perturbed delay differential equation as follows:

$$\begin{aligned}&f(s,t)=\Psi(s,t),\quad (s,t)\in[0,1]\times[-\tau,0],\\&\frac{\partial f(s,t)}{\partial t}-\varepsilon\frac{\partial^2 f(s,t)}{\partial s^2}=-e^{-0.05}f(s,t-\tau)+F(s,t),\ \ (s,t)\in[0,1]\times(0,2],\\&f(0,t)=0,\quad f(1,t)=0,\quad t\in[0,2],\end{aligned}$$

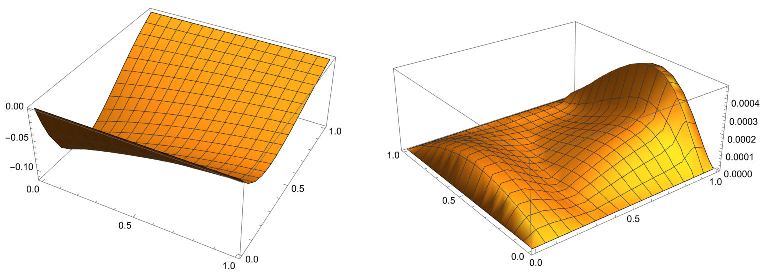
where $\tau=0.05$, and the source function is provided by

$$F(s,t)=e^{-\left(t+s/\sqrt{\varepsilon}\right)}\left(-s(s-1)+2(2s-1)\sqrt{\varepsilon}-2\varepsilon\right).$$

The initial data is given by $\Psi(s,t)$ which can be calculated from the exact solution

$$f(s,t) = s(s-1)e^{-\left(t+s/\sqrt{\varepsilon}\right)}.$$

The profiles of the approximate solution and the absolute errors when $n = 64$ with $\epsilon = 2^{-2}$ are shown in Figure 1.



a) The approximating solution, b) The absolute error

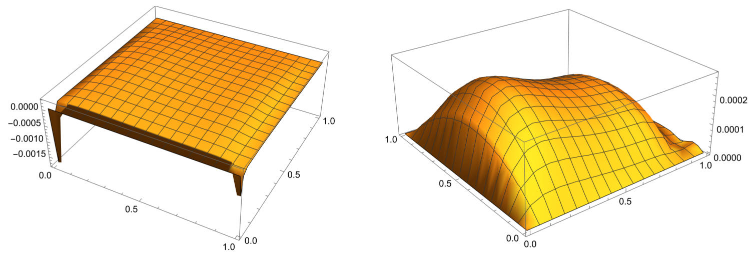
Figure 1. Example 1 – a) the approximating solution and b) the absolute error with $\epsilon = 2^{-2}$ and $\tau = 0.05$.

Table 1 is listed the absolute errors regarding different values of the singularity perturbed parameter ϵ and different values of spatial points n.

Table 1. Errors and convergence orders of adaptive RKM for Example 1.

ϵ	n	$\|e_n\|_0$	order	$\|e_n\|_{1,t}$	order	$\|e_n\|_{1,s}$	order	$\|e_n\|_{2,s}$	order
2^{-2}	16	1.49E-3		6.85E-3		4.83E-3		2.47E-2	
	64	3.36E-4	2.14	2.25E-3	1.60	1.23E-3	1.97	6.25E-3	1.98
	256	7.74E-5	2.13	6.39E-4	1.84	3.01E-4	1.99	1.57E-3	1.99
	1024	1.85E-5	2.05	1.68E-4	1.93	7.51E-5	1.99	3.93E-4	2.00
2^{-4}	16	1.55E-3		7.22E-3		8.39E-3		8.61E-2	
	64	3.40E-4	2.19	1.80E-3	2.00	1.48E-3	2.50	2.07E-2	2.06
	256	7.65E-5	2.15	4.55E-4	1.98	2.83E-3	2.38	4.97E-3	2.05
	1024	1.82E-5	2.08	1.14E-4	1.99	6.84E-4	2.04	1.21E-3	2.04
2^{-6}	16	5.39E-3		2.01E-2		2.79E-2		2.65E-1	
	64	1.26E-3	2.07	5.19E-3	1.95	7.21E-3	1.95	7.01E-2	1.91
	256	3.04E-4	2.05	1.30E-3	2.00	1.83E-3	1.98	1.81E-2	1.95
	1024	7.51E-5	2.02	3.24E-4	1.99	4.58E-4	2.00	4.62E-3	1.97

It can be shown clearly that the proposed numerical method converges with orders of $\mathcal{O}(h^2)$ under $\mathbb{L}^2$ norm, H^1 seminorm and H^2 seminorm, which is consistent with traditional RKM. The computational accuracy is decreasing when ϵ is getting smaller. Figure 2 shows the the profiles of the approximated solution and the absolute errors when $n = 256$ with $\epsilon = 2^{-8}$. As we can see from Figure 2, the proposed algorithm can handle $\epsilon = 2^{-8}$ with fairly accurate approximations.



a) The approximating solution, b) The absolute error

Figure 2. Example 1 – a) the approximating solution and b) the absolute error with $\epsilon = 2^{-8}$ and $\tau = 0.05$.

Example 2. Let us examine the equation as follows:

$$\begin{aligned} &f(s,t) = \Psi(s,t), \quad (s,t) \in [-\tau, 0] \times [0,1], \\ &\frac{\partial f(s,t)}{\partial t} - \varepsilon \frac{\partial^2 f(s,t)}{\partial s^2} = -2f(s,t-\tau) + F(s,t), \quad (s,t) \in [0,1] \times (0,2], \\ &f(0,t) = 0, \quad f(1,t) = 0, \quad t \in [0,2], \end{aligned}$$

where $\tau = 0.01$, and the source function is provided by

$$F(s,t) = e^{-\left(t+s/\sqrt{\varepsilon}\right)} \left(2s(s-1)^2(-1+e^{0.01}) + 2(3s^2-4s+1)\sqrt{\varepsilon} - 2(s-2)\varepsilon\right).$$

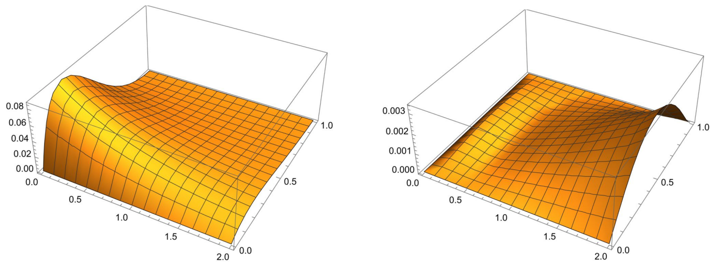
The initial data is given by $\Psi(s,t)$ which can be calculated from the exact solution

$$f(s,t) = s(s-1)^2 e^{-\left(t+s/\sqrt{\varepsilon}\right)}.$$

Table 2. Errors and convergence orders of adaptive RKM for Example 2.

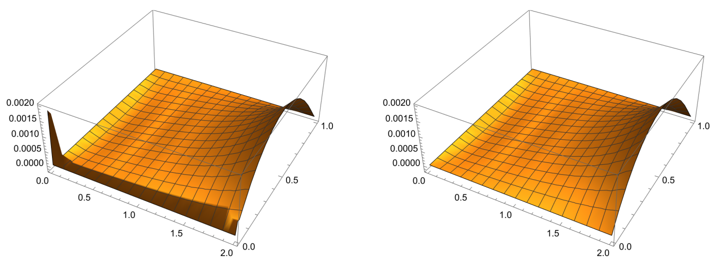
ϵ	n	$\|e_n\|_0$	order	$\|e_n\|_{1,t}$	order	$\|e_n\|_{1,s}$	order	$\|e_n\|_{2,s}$	order
2^{-2}	16	9.87E-3		2.81E-2		3.32E-2		1.08E-1	
	64	2.82E-3	1.68	7.98E-3	1.81	9.45E-3	1.81	3.13E-2	1.79
	256	7.95E-4	1.83	2.07E-3	1.94	2.51E-3	1.92	8.65E-3	1.86
	1024	2.05E-4	1.95	5.22E-4	1.99	6.54E-4	1.93	2.23E-3	1.96
2^{-4}	16	2.16E-2		8.01E-2		8.64E-2		5.51E-1	
	64	6.26E-3	1.79	2.25E-2	1.83	2.53E-2	1.77	1.58E-1	1.80
	256	1.64E-3	1.93	6.02E-3	1.90	6.90E-3	1.88	4.29E-2	1.88
	1024	4.21E-4	1.97	1.55E-3	1.96	1.85E-3	1.90	1.12E-2	1.94
2^{-6}	16	5.24E-2		2.62E-1		2.59E-1		2.38E-0	
	64	1.49E-2	1.82	7.34E-2	1.84	7.10E-2	1.87	6.74E-1	1.82
	256	4.05E-3	1.88	1.98E-2	1.89	1.89E-2	1.91	1.79E-1	1.91
	1024	1.04E-3	1.96	5.15E-3	1.94	4.83E-3	1.97	4.71E-2	1.93

Listed in Table 2 are numerical results of Example 2 obtained by our proposed RKM. By applying the adaptive strategies, we obtain a similar convergence results as Example 1. The profiles of the approximated solution and the absolute errors when with $\epsilon = 2^{-2}(n = 64)$ and $\epsilon = 2^{-8}(n = 256)$ are shown Figures 3 and 4, respectively. As ϵ gets smaller, the accuracy remains at the similar order of magnitudes. Nevertheless, our adaptive RKM improve the accuracy compared with the traditional RKM.



a) The approximating solution, b) The absolute error

Figure 3. Example 2 – a) the approximating solution and b) the absolute error with $\epsilon = 2^{-2}$ and $\tau = 0.01$.



a) The approximating solution, b) The absolute error

Figure 4. Example 2 – a) the approximating solution and b) the absolute error with 2^{-8} and $\tau = 0.01$.

Example 3. Let us compare the equation in [3] as follows:

$$\begin{aligned}
&f(s,t) = \Psi(s,t), \quad (s,t) \in [-\tau, 0] \times [0,1],\\
&\frac{\partial f(s,t)}{\partial t} - \varepsilon \frac{\partial^2 f(s,t)}{\partial s^2} = -2e^{-1} f(s,t-1) + F(s,t), \quad (s,t) \in [0,1] \times (0,2],\\
&f(0,t) = e^{-1}, \quad f(1,t) = e^{-\left(t+1/\sqrt{\varepsilon}\right)}, \quad t \in [0,2].
\end{aligned}$$

The initial date is given by $\Psi(s,t)$ which can be calculated from the exact solution

$$f(s,t) = e^{-\left(t+s/\sqrt{\varepsilon}\right)}.$$

Listed in Table 3 are numerical results of Example 3 obtained by our proposed RKM and the finite difference methods in [3]. From the Table, we can see that our RKM method is litte bit more accurate than the method in [3]. This also shows that the RKM proposed in this paper is meaningful.

Table 3. The comparision of maximum errors for Example 3.

ϵ	n	parameter-robust FDMs in [3]	our proposed RKM
2^{-6}	64	2.158E-3	1.335E-3
	256	5.138E-4	3.179E-4
	1024	1.268E-4	7.948E-5
2^{-8}	64	2.628E-3	1.594E-3
	256	5.449E-4	3.785E-4
	1024	1.287E-4	9.463E-5
2^{-14}	64	4.718E-3	2.947E-3
	256	8.212E-4	7.017E-4
	1024	1.576E-4	1.754E-4

6 Conclusions

In this post, a significant method was proposed by us that using RK-spaces and collocation method to solve delay parabolic PDEs with singular perturbation. We defined three basic RK-spaces with their inner product and norms. Furthermore, an approximated solution to the delay parabolic PDEs with singular perturbation were approximated by the RK-space $\mathbb{W}_{(3,2)(\Omega)}$. In addition, we verified that the exact solution is uniformly convergent by the approximated solution. Error estimates for the presented numerical algorithm were established.

All the discussions and proofs are based on $[0,1]$ in one dimensional space. However, those results can be easily extended to other closed interval in $\mathcal{R}$. Furthermore, the absolute errors of the approximated solution is in the order of T/n which can be understood as the time step size in our numerical algorithm. Notice that we do not have any special time discretization in our algorithm. In other words, the time domain is treated the same way as the spatial domain, which is much easier than other traditional methods that use finite different scheme for time discretization and another spatial discretization scheme.

Acknowledgements

This study was supported by
1. National Natural Science Funds of China under Grant No.12101164;
2. General support from Heilongjiang Postdoctoral Fund LBH-Z22256;
3. Heilongjiang Higher Education Reform Project SJGZ20210095.

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