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Homotopy Method for a General Multiobjective Programming Problem

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Abstract In this paper, we present a combined homotopy interior-point method for a general multiobjective programming problem. The algorithm generated by this method associated to Karush–Kuhn–Tucker points of the multiobjective programming problem is proved to be globally convergent under some basic assumptions.

Keywords Multiobjective programming problems · Homotopy methods · KKT conditions · Efficient solutions

1 Introduction

We consider the following multiobjective programming problem:

$$\text{(MOP)} \qquad \begin{aligned} \min \quad & f(x), \\ \text{s.t.} \quad & g(x) \leqq 0, \\ & h(x) = 0, \end{aligned}$$

where $f = (f_1, f_2, \ldots, f_p)^T : R^n \to R^p$, $g = (g_1, g_2, \ldots, g_m)^T : R^n \to R^m$ and $h = (h_1, h_2, \ldots, h_s)^T : R^n \to R^s$ are twice continuously differentiable functions.

It is well known that, if x is an efficient solution of (MOP), under some constraint qualifications [Kuhn and Tucker constraint qualification (see [1]) or Abadie constraint

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qualification (see [2])], then the following Karush–Kuhn–Tucker (KKT) condition at x for (MOP) holds (see [3, 4]):

$$\nabla f(x)\lambda + \nabla g(x)u + \nabla h(x)v = 0,$$
$$Ug(x) = 0,$$

where $\lambda \in \mathbb{R}^p_+ \setminus \{0\}$, $u \in \mathbb{R}^m_+$, $v \in \mathbb{R}^s$, and $U = \operatorname{diag}\{u_1, u_2, \dots, u_m\}$. We say that x is a KKT point of (MOP) if it satisfies the KKT condition.

Since the remarkable papers of Kellogg et al. [5] and Chow et al. [6] have been published, more and more attention has been paid to the homotopy method. As a globally convergent method, the homotopy method (or path-following method) now becomes an important tool for numerically solving nonlinear problems including nonlinear mathematical programming and complementary problems (see [7, 8]).

In 1988, Megiddo [9] and Kojima et al. [10] discovered that the Karmakar interior point method was a kind of path-following method for solving linear programming. Since then, the interior path-following method has been generalized to convex programming, and becomes one of the main method for solving mathematical programming problems. Among most interior methods, one of the main idea is numerically tracing the center path generated by the optimal solution set of the so-called logarithmic barrier function. Usually, the strict convexity of the logarithmic barrier function or nonemptiness and boundedness of the solution set [11, 12] is needed. In 1997, Lin, Yu and Feng [13] presented a new interior point method—combined homotopy interior point method (CHIP method)—for convex nonlinear programming without such assumptions. Subsequently, Lin, Li and Yu [14] generalized the CHIP method to general nonlinear programming where, instead of the convexity conditions, they used a more general normal cone condition.

Recently Lin, Zhu and Sheng [15] generalized the CHIP method to convex multiobjective programming with only inequality constraints,

$$(\text{CMOP1}) \qquad \begin{aligned} &\min \quad f(x),\\ &\text{s.t.} \quad g(x) \leqq 0, \end{aligned}$$

Instead of (CMOP1), they considered an associated nonconvex nonlinear scalar optimization problem in the variables (x, λ) as following:

$$(\text{CMOP2}(\lambda)) \qquad \begin{aligned} &\min \quad \lambda^T f(x),\\ &\text{s.t.} \quad g(x) \leqq 0,\\ &\qquad \sum_{i=1}^{p} \lambda_i = 1,\\ &\qquad \lambda_i \geqslant 0, \quad x \in \mathbb{R}^n, \end{aligned}$$

and constructed the following homotopy mapping:

$$H(t,\omega^0,\omega)=\begin{bmatrix}(1-t)(\nabla f(x)\lambda+\nabla g(x)z)+t(x-x^0)\\(1-t)(f(x)+w)-ye+t(\lambda-\lambda^0)\\1-\sum_{i=1}^p\lambda_i\\Zg(x)-tZ^0g(x^0)\\W\lambda-tW^0\lambda^0\end{bmatrix},$$

where $Z:=\operatorname{diag}\{z_1,\dots,z_m\}$, $W:=\operatorname{diag}\{w_1,\dots,w_p\}$, for solving the KKT system (CMOP2(λ)),

$$\begin{aligned}&\nabla f(x)\lambda+\nabla g(x)z=0,\\&f(x)-ye-w=0,\\&1-\sum_{i=1}^p\lambda_i=0,\\&Zg(x)=0,\quad g(x)\leqq 0,\quad z\geqq 0,\\&W\lambda=0,\quad \lambda\geqq 0,\quad w\geqq 0.\end{aligned}$$

The purpose of this paper is to generalize the combined homotopy interior point method to the general multiobjective programming problem (MOP). The paper is organized as following. In Sect. 2, we recall some notations and preliminaries results, and we construct a new combined homotopy mapping which is related directly to the KKT system of (MOP) and is much simpler than that one given in [15]. In Sect. 3, we prove the existence and convergence of a smooth homotopy path from almost any interior initial point ω^0 to a solution of the KKT system of (MOP) under so-called normal cone condition of constraints. A numerical algorithm is given. In Sect. 4, we relax the boundedness condition concerning the feasible set and obtain a similar result for the convex multiobjective program problem.

2 Some Definitions and Properties

Let $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the nonnegative and positive orthant of $\mathbb{R}^n$, respectively. For any two vectors $y=(y_1,y_2,\dots,y_n)$ and $z=(z_1,z_2,\dots,z_n)$ in $\mathbb{R}^n$, we use the following conventions:

$$\begin{aligned}&y=z,\quad \text{iff } y_i=z_i,\ i=1,2,\dots,n;\\&y\leqq z,\quad \text{iff } y_i\leqslant z_i,\ i=1,2,\dots,n;\\&y<z,\quad \text{iff } y_i<z_i,\ i=1,2,\dots,n;\\&y\leq z,\quad \text{iff } y_i\leqslant z_i,\ \text{and } y\neq z,\ i=1,2,\dots,n.\end{aligned}$$

Suppose that $f=(f_1,f_2,\dots,f_p)^T:R^n\to R^p$, $g=(g_1,g_2,\dots,g_m)^T:R^n\to R^m$ and $h=(h_1,h_2,\dots,h_s)^T:R^n\to R^s$ are twice continuously differentiable functions. Let

$$\begin{aligned}
&\Omega=\{x\in\mathbb{R}^n: g(x)<0, h(x)=0\},\\
&\Omega_1=\{x\in\mathbb{R}^n: g(x)<0\},\\
&\Lambda^{++}=\left\{\lambda\in\mathbb{R}^p_{++}:\sum_{i=1}^p\lambda_i=1\right\},\\
&I:=\{1,2,\dots,m\},\quad J:=\{1,2,\dots,s\},
\end{aligned}$$

and let

$$I(x):=\{i\in I: g_i(x)=0\}$$

denote the index set of the active inequality constraints at a given point,

$$x\in\bar{\Omega}=\{x\in\mathbb{R}^n: g(x)\leqq 0, h(x)=0\}.$$

In the literature, solutions for a multiobjective programming problem are referred to variously as efficient, Pareto-optimal, and nondominated solutions. In this paper we shall refer to a solution of a multiobjective programming problem as an efficient solution.

Definition 2.1 A point $x\in\bar{\Omega}$ is said to be an efficient solution to problem (MOP) if there is no $y\in\bar{\Omega}$ such that $f(y)\le f(x)$ holds.

Definition 2.2 (See [16]) Let $C\subset\mathbb{R}^n$ be a closed subset and let $x_0\in C$. A vector $v\in\mathbb{R}^n$ is called a strict normal vector to C at x_0, if $\langle v,\ y-x_0\rangle\le 0$, for all $y\to x_0$ and $y\in C$, i.e.,

$$\limsup_{\substack{y\to x_0\\ y\in C\setminus\{x_0\}}}\langle v,\ y-x_0\rangle\le 0.$$

We denote by $\widehat{N}_C(x_0)$ the set of strict normal vectors to C at x_0.

Lemma 2.1 (See [16]) *Let* $x\in\bar{\Omega}$. *If the vectors* $\{\nabla g_i(x), i\in I(x), \nabla h_j(x), j\in J\}$ *are linearly independent, then*

$$\widehat{N}_{\bar{\Omega}}(x)=\left\{\sum_{i\in I(x)}u_i\nabla g_i(x)+\sum_{j\in J}v_j\nabla h_j(x), u_i\ge 0, i\in I(x), v\in\mathbb{R}^s\right\}.$$

The following three basic conditions are commonly used in the literature:

(A1) Ω is nonempty and bounded;
(A2) $\forall x\in\bar{\Omega}$, the vectors $\{\nabla g_i(x), i\in I(x), \nabla h_j(x), j\in J\}$ are linearly independent;
(A3) $\forall x\in\bar{\Omega}$, $\{x+\widehat{N}_{\bar{\Omega}}(x)\}\cap\bar{\Omega}=\{x\}$.

Observe that the normal cone condition (A_3) is a generalization of the convexity, i.e., if $\bar{\Omega}$ is a convex set, then the condition (A_3) is satisfied automatically. In the case when (MOP) is convex, we can relax condition of the bounded feasible set; see the fourth section for more details.

Let $x \in \bar{\Omega}$ be a KKT point of (MOP). Then, there exist $\lambda \in \mathbb{R}^p_+ \setminus \{0\}$, $u \in \mathbb{R}^m_+$, $v \in \mathbb{R}^s$, and $U = \mathrm{diag}\{u_1, u_2, \ldots, u_m\}$ such that

$$\nabla f(x)\lambda + \nabla g(x)u + \nabla h(x)v = 0,$$
$$Ug(x) = 0.$$

Since $\lambda \geq 0$, without loss of generality we can assume that $\sum_{i=1}^p \lambda_i = 1$. Our aim is to find $\omega = (x, \lambda, u, v) \in \bar{\Omega} \times \mathbb{R}^{p+m}_+ \times \mathbb{R}^s$ such that

$$\nabla f(x)\lambda + \nabla g(x)u + \nabla h(x)v = 0, \tag{1a}$$
$$Ug(x) = 0, \tag{1b}$$
$$1 - \sum_{i=1}^p \lambda_i = 0. \tag{1c}$$

We construct a homotopy as follows

$$H(\omega, \omega^0, \mu) = \begin{bmatrix} (1-\mu)(\nabla f(x)\lambda + \nabla g(x)u) + \nabla h(x)v + \mu(x - x^0) \\ h(x) \\ Ug(x) - \mu U^0 g(x^0) \\ (1-\mu)(1 - \sum_{i=1}^p \lambda_i)e - \mu(\lambda - \lambda^0) \end{bmatrix} = 0, \tag{2}$$

where $\omega^0 = (x^0, \lambda^0, u^0, v^0) \in \Omega \times \Lambda^{++} \times \mathbb{R}^m_{++} \times \{0\}$, $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^p$, $\omega = (x, \lambda, u, v) \in \bar{\Omega} \times \mathbb{R}^{p+m}_{++} \times \mathbb{R}^s$ and $\mu \in [0, 1]$.

When $\mu = 1$, the homotopy equation (2) becomes

$$\nabla h(x)v + x - x^0 = 0, \tag{3a}$$
$$h(x) = 0, \tag{3b}$$
$$Ug(x) - U^0 g(x^0) = 0, \tag{3c}$$
$$\lambda - \lambda^0 = 0. \tag{3d}$$

By Lemma 2.1 and the condition (A_3), (3a) implies that $\nabla h(x)v + x = x^0 \in \{x + \widehat{N}_{\bar{\Omega}}(x)\} \cap \bar{\Omega} = \{x\}$. Hence $v = v^0 = 0$ by the condition (A_2). Since $g(x^0) < 0$ and $x = x^0$, (3c) implies that $u = u^0$. Thus $\omega = \omega^0$. That is, the equation $H(\omega, \omega^0, 1) = 0$ with respect to ω has only one solution $\omega = \omega^0$.

As $\mu = 0$, $H(\omega, \omega^0, \mu) = 0$ turns to problem (1a–1c). For a given ω^0, rewrite $H(\omega, \omega^0, \mu)$ as $H_{\omega^0}(\omega, \mu)$. The zero-point set of H_{ω^0} is

$$H^{-1}_{\omega^0}(0) = \{(\omega, \mu) \in \Omega \times \mathbb{R}^{p+m}_{++} \times \mathbb{R}^s \times (0, 1] : \quad H(\omega, \omega^0, \mu) = 0\}.$$

Since $H(\omega^0, \omega^0, 1) = 0$, we have $H^{-1}_{\omega^0}(0) \neq \emptyset$.

Let us recall some basic definitions and results from differential topology. For the definitions of C^r differential manifold, submanifold, and C^r differential manifold with boundary, we refer the reader to [17, 18]. Clearly Ω_1 is a n-dimensional differential manifold, Ω is a $(n-s)$-dimensional differential manifold and $\Omega \times \mathbb{R}_{++}^{p+m} \times \mathbb{R}^s \times (0,1]$ is a manifold with boundary $\Omega \times \mathbb{R}_{++}^{p+m} \times \mathbb{R}^s \times \{1\}$.

Definition 2.3 Let M, N be differential manifolds with $\dim N = p$ and let $H : M \to N$ be a differentiable mapping. If

$$\operatorname{rank}\left[\frac{\partial H(x)}{\partial x}\right] = p, \quad \forall x \in H^{-1}(y),$$

we say that $y \in N$ is a regular value of H and $x \in M$ is a regular point. Given a curve $\Gamma \subset H^{-1}(y)$, if every $x \in \Gamma$ is a regular point, then we say that Γ is a regular path.

Lemma 2.2 (Parametric Form of the Sard Theorem on a Manifold with Boundary) *Let Λ and N be differential manifolds of dimension q and p, respectively, and let M be a m-dimensional differential manifold with boundary. Suppose that $F : \Lambda \times M \to N$ is a C^r mapping, where $r > \max\{0, m-p\}$. If $0 \in N$ is a regular value of F and ∂F, then for almost all $\lambda \in \Lambda$, 0 is a regular value of $F_\lambda = F(\lambda, \cdot)$ and ∂F_λ, where ∂F, ∂F_λ denote the restriction of F and F_λ to $\Lambda \times \partial M$ and ∂M, respectively.*

This lemma is a special case of the transversality theorem (Theorem 5.7 in [18]).

Lemma 2.3 (Inverse Image Theorem; See [17, 18]) *Suppose that M is an m-dimensional C^r differential manifold with boundary, N is a p-dimensional C^r differential manifold, $r \geqslant 1$, and $F : M \to N$ is a C^r map. If $q \in N$ is a regular value of F and ∂F, then either $S = F^{-1}(q)$ is empty or a $(m-p)$-dimensional submanifold, and*

$$\partial S = S \cap \partial M.$$

Lemma 2.4 (Classification Theorem of One-Dimensional Manifold with Boundary; See [17]) *Each connected part of a one-dimensional manifold with boundary is homeomorphic either to a unit circle or to a unit interval.*

3 Main Results

In this section, we shall present the main results of the paper.

Theorem 3.1 *Suppose that $\Omega \neq \emptyset$ and condition (A_2) hold. Then for almost all initial points $\omega^0 \in \Omega \times \Lambda^{++} \times \mathbb{R}_{++}^m \times \{0\}$, 0 is a regular value of H_{ω^0} and $H_{\omega^0}^{-1}(0)$ consists of some smooth curves. Among them, a smooth curve, say Γ_{ω^0}, starts from $(\omega^0, 1)$.*

Proof For any $(\omega, \omega^0, \mu) \in H^{-1}(0)$,

$$\frac{\partial H(\omega, \omega^0, \mu)}{\partial(x, x^0, \lambda^0, u^0)} = \begin{bmatrix} Q(x) & -\mu I & 0 & 0 \\ \nabla h(x) & 0 & 0 & 0 \\ U\nabla g(x) & -\mu U^0 \nabla g(x^0) & 0 & -\mu \mathrm{diag}(g(x^0)) \\ 0 & 0 & \mu I & 0 \end{bmatrix},$$

where

$$Q(x) = (1-\mu)\left(\sum_{i=1}^{p} \lambda_i \nabla^2 f_i(x) + \sum_{j=1}^{m} u_j \nabla^2 g_j(x)\right) + \sum_{k=1}^{s} v_k \nabla^2 h_k(x) + \mu I.$$

Because $x, x^0 \in \Omega$ and $\mu \in (0, 1]$, by the regularity condition (A_2), we obtain that

$$\mathrm{rank}\left[\frac{\partial H(\omega,\ \omega^0,\ \mu)}{\partial(x, x^0, \lambda^0, u^0)}\right] = n + p + m + s.$$

Thus both the Jacobian matrices $\frac{\partial H(\omega,\ \omega^0,\ \mu)}{\partial(\omega,\ \omega^0,\ \mu)}$ and $\frac{\partial H(\omega,\omega^0,1)}{\partial(\omega,\omega^0)}$ are of full row rank, that is, 0 is a regular value of H and ∂H. By Lemma 2.2, for almost all $\omega^0 \in \Omega \times \Lambda^{++} \times \mathbb{R}^m_{++} \times \{0\}$, 0 is a regular value of H_{ω^0}. By Lemma 2.3, $H^{-1}_{\omega^0}(0)$ consists of some smooth curves. Since $H(\omega^0, \omega^0, 1) = 0$, there must be a smooth curve, denoted by Γ_{ω^0}, that starts from $(\omega^0, 1)$. □

Theorem 3.2 *Let $\Omega \neq \emptyset$ and let Assumption* (A2) *hold. For a given $\omega^0 \in \Omega \times \Lambda^{++} \times \mathbb{R}^m_{++} \times \mathbb{R}^s$, if* 0 *is a regular value of H_{ω^0}, then the projection of the smooth curve Γ_{ω^0} on the λ component is bounded.*

Proof Suppose that the conclusion does not hold. Since (0,1] is bounded, there exists a sequence $\{(\omega^k, \mu_k)\} \subset \Gamma_{\omega^0}$ such that,

$$\mu_k \to \mu_*, \quad \|\lambda^k\| \to +\infty, \quad k \to \infty.$$

From the homotopy equation (2), we have

$$(1-\mu_k)\left(1 - \sum_{i=1}^{p} \lambda_i^k\right) e - \mu_k(\lambda^k - \lambda^0) = 0. \tag{4}$$

That is,

$$\begin{bmatrix} 1-\mu_k \\ 1-\mu_k \\ \vdots \\ 1-\mu_k \end{bmatrix} - \begin{bmatrix} \lambda_1^k + (1-\mu_k)\sum_{i=2}^{p} \lambda_i^k \\ \lambda_2^k + (1-\mu_k)\sum\limits_{i\neq 2} \lambda_i^k \\ \vdots \\ \lambda_p^k + (1-\mu_k)\sum\limits_{i\neq p} \lambda_i^k \end{bmatrix} + \mu_k \begin{bmatrix} \lambda_1^0 \\ \lambda_2^0 \\ \vdots \\ \lambda_p^0 \end{bmatrix} = 0. \tag{5}$$

Let

$$I = \{j \in \{1, 2, \dots, p\} : \lim_{k\to\infty} \lambda_j^k = \infty\}.$$

The hypothesis implies $I \neq \emptyset$. Since $\mu_k \to \mu_* \in [0, 1]$ and $\lambda^k \geq 0$, it follows that the second part in the left-hand side of some equation of (5) tends to infinity as $k \to +\infty$, but the first and third parts are bounded. This is a contradiction. Thus the projection of the smooth curve Γ_{ω^0} on the λ component is bounded. □

Theorem 3.3 (Boundedness) *Let Assumptions* (A1–A3) *hold. Then, for a given* $\omega^0 \in \Omega \times \Lambda^{++} \times \mathbb{R}^m_{++} \times \mathbb{R}^s$, *if* 0 *is a regular value of* H_{ω^0}, *then* Γ_{ω^0} *is a bounded curve.*

Proof Assume that Γ_{ω^0} is an unbounded curve. Then, by the boundedness of Ω and Theorem 3.2, there exists a sequence $\{(\omega^k, \mu_k)\} \subset \Gamma_{\omega^0}$ such that,

$$x^k \to x^*, \quad \mu_k \to \mu_*, \quad \lambda^k \to \lambda^*, \quad \|(u^k, v^k)\| \to +\infty, \quad k \to \infty.$$

From the homotopy equation (2), we have

$$(1-\mu_k)(\nabla f(x^k)\lambda^k + \nabla g(x^k)u^k) + \nabla h(x^k)v^k + \mu_k(x^k - x^0) = 0, \tag{6}$$

$$h(x^k) = 0, \tag{7}$$

$$U^k \times g(x^k) - \mu_k U^0 \times g(x^0) = 0. \tag{8}$$

Let

$$I_2(x^*) = \left\{j \in J : \lim_{k\to\infty} v_j^k = \infty\right\}, \; I_3(x^*) = \left\{j \in I : \lim_{k\to\infty} u_j^k = \infty\right\}.$$

If $I_2(x^*) \neq \emptyset$, rewrite (6) as

$$(1-\mu_k)\left[\nabla f(x^k)\lambda^k + \sum_{j\notin I_3(x^*)} u_j^k \nabla g_j(x^k)\right] + \left[\nabla h(x^k)v^k + (1-\mu_k)\sum_{j\in I_3(x^*)} u_j^k \nabla g_j(x^k)\right] + \mu_k(x^k - x^0) = 0. \tag{9}$$

It follows from (8) that $I_3(x^*) \subset I(x^*)$. Since $I_2(x^*) \neq \emptyset$ and (A2) holds, the second part in the left-hand side of (9) tends to infinity as $k \to \infty$, but the other two parts are bounded, this is impossible. Thus $I_2(x^*) = \emptyset$. We can assume that $v^k \to v^*(k \to \infty)$. At the same time we have $I_3(x^*) \neq \emptyset$.

(i) When $\mu_* = 1$, rewrite (6) as

$$\nabla h(x^k)v^k + \sum_{j\in I(x^*)} (1-\mu_k)u_j^k \nabla g_j(x^k) + x^k - x^0 = (1-\mu_k)\left[-\sum_{j\notin I(x^*)} u_j^k \nabla g_j(x^k) - \nabla f(x^k)\lambda^k + x^k - x^0\right].$$

Let $k\to\infty$; since (A2) holds, the above equation becomes

$$x^* + \sum_{j\in I(x^*)} \lim_{k\to\infty}[(1-\mu_k)u_j^k]\nabla g_j(x^*) + \nabla h(x^*)v^* = x^0 \in \Omega. \tag{10}$$

From the normal cone condition (A3), (10) implies that $x^* = x^0$. This is impossible.

(ii) When $\mu_* \in [0,1)$, we rewrite (6) as

$$\begin{aligned}(1-\mu_k)\Bigg[\nabla f(x^k)\lambda^k + \sum_{j\notin I_3(x^*)} u_j^k \nabla g_j(x^k)\Bigg]\\ + \nabla h(x^k)v^k + (1-\mu_k)\sum_{j\in I_3(x^*)} u_j^k\nabla g_j(x^k) + \mu_k(x^k - x^0) = 0. \end{aligned}\tag{11}$$

From the fact that $\lambda^k, v^k, u_j^k (j\notin I_3(x^*))$ are bounded, $\mu_*\in[0,1)$, and $I_3(x^*)\subset I(x^*)$, by (A2), the third part in the left-hand side in (11) tends to infinity as $k\to\infty$, but the other three parts are bounded, this is also impossible. Therefore, Γ_{ω^0} is a bounded curve. □

Theorem 3.4 (Convergence of the Method) *Let Assumptions* (A1–A3) *hold. Then, for almost all* $\omega^0\in\Omega\times\Lambda^{++}\times\mathbb{R}^m_{++}\times\mathbb{R}^s$, *the zero-point set* $H^{-1}_{\omega^0}(0)$ *of the homotopy map* (2) *contains a smooth curve* $\Gamma_{\omega^0}\subset\bar{\Omega}\times\mathbb{R}^{p+m}_+\times(0,1]$, *which starts from* $(\omega^0,1)$. *As* $\mu\to 0$, *the limit set* $T\times\{0\}\subset\bar{\Omega}\times\mathbb{R}^{p+m}_+\times\{0\}$ *of* Γ_{ω^0} *is nonempty and every point in* T *is a solution of* (2).

Proof By Theorem 3.1 and Theorem 3.3, the existence of Γ_{ω^0} is obtained. It remains to show that the limit set $T\times\{0\}\subset\bar{\Omega}\times\mathbb{R}^{p+m}_+\times\{0\}$ of Γ_{ω^0} is nonempty as $\mu\to 0$. By Lemma 2.4, Γ_{ω^0} is diffeomorphic to a unit circle or the unit interval. Since Γ_{ω^0} is a bounded curve, if we let the variable s be the arclength of Γ_{ω^0}, then the parametric form of Γ_{ω^0} is

$$(\omega(s),\mu(s)),\ \ 0\le s\le s_0, \qquad (\omega(0),\mu(0)) = (\omega^0,1).$$

It is clear that the unit tangent vector $\xi^0=(\xi_1^0,\xi_2^0)^T\in\mathbb{R}^{n+p+m+s+1}$ of Γ_{ω^0} at $(\omega^0,1)$ satisfies

$$DH_{\omega^0}(\omega^0,1)\xi^0 = 0,$$

where $\xi_1^0\in\mathbb{R}^{n+p+m+s}$, $\xi_2^0\in\mathbb{R}$. Because

$$\begin{aligned} DH_{\omega^0}(\omega^0,1) &= \begin{bmatrix} I & 0 & 0 & \nabla h(x^0) & \nabla f(x^0)\lambda^0 - \nabla g(x^0)u^0 \\ \nabla h(x^0) & 0 & 0 & 0 & 0 \\ U^0\nabla g(x^0) & 0 & \operatorname{diag}(g(x^0)) & 0 & -U^0 g(x^0) \\ 0 & -I & 0 & 0 & 0 \end{bmatrix} \\ &:= (M_1, M_2), \end{aligned}$$

where $M_1 \in \mathbb{R}^{(n+p+m+s)\times(n+p+m+s)}$, $M_2 \in \mathbb{R}^{n+p+m+s}$, and because M_1 is nonsingular and $M_2 \neq 0$,

$$\xi^0 = (1 + \|M_1^{-1} M_2\|^2)^{-\frac{1}{2}} \begin{pmatrix} -M_1^{-1} M_2 \\ 1 \end{pmatrix} \quad \text{and} \quad \xi_2^0 \neq 1.$$

Thus Γ_{ω^0} is not tangent to the plane $\mu = 1$ at $(\omega^0, 1)$. Because the equation $H(\omega, \omega^0, 1) = 0$ has only one solution $\omega = \omega^0$, it follows that Γ_{ω^0} cannot be diffeomorphic to a unit circle, but a unit interval. Thus it has a limit point while $\mu \to 0$, we assume that $(\bar{\omega}, \bar{\mu})$ is a limit point. Then $(\bar{\omega}, \bar{\mu})$ in $\partial(\Omega_1 \times \mathbb{R}_{++}^{p+m} \times \mathbb{R}^s \times (0,1])$. In fact, if $(\bar{\omega}, \bar{\mu}) \in \Omega^1 := \Omega_1 \times \mathbb{R}_{++}^{p+m} \times \mathbb{R}^s \times (0,1)$, since 0 is a regular value of $H_{\omega^0} : \Omega^1 \to \mathbb{R}^{n+p+m+s}$ and $(\bar{\omega}, \bar{\mu}) \in H_{\omega^0}^{-1}(0)$, the Jacobian matrix of H at $(\bar{\omega}, \bar{\mu})$ is of full row rank. By the implicit function theorem, Γ_{ω^0} can be extended at $(\bar{\omega}, \bar{\mu})$. This contradicts the fact that $(\bar{\omega}, \bar{\mu})$ is a limit point of Γ_{ω^0}. Let $(\bar{\omega}, \bar{\mu}) = (\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{\mu})$. Then $(\bar{\omega}, \bar{\mu}) \in \partial(\Omega_1 \times \mathbb{R}_{++}^{p+m} \times \mathbb{R}^s \times (0,1])$ and only the following three case are possible:

(a) $(\bar{\omega}, \bar{\mu}) \in \bar{\Omega}_1 \times \mathbb{R}_{+}^{p+m} \times \mathbb{R}^s \times \{1\}$;
(b) $(\bar{\omega}, \bar{\mu}) \in \partial(\Omega_1 \times \mathbb{R}_{++}^{p+m}) \times \mathbb{R}^s \times (0,1)$;
(c) $(\bar{\omega}, \bar{\mu}) \in \bar{\Omega}_1 \times \mathbb{R}_{+}^{p+m} \times \mathbb{R}^s \times \{0\}$.

Because the equation $H(\omega, \omega^0, 1) = 0$ has only one solution $\omega = \omega^0$, the case (a) is impossible.

In case (b), we first prove that $\bar{u} \notin \partial\mathbb{R}_{++}^m$.

Indeed, if there exists $j_0 \in I$ such that $\bar{u}_{j_0} = 0$, then there exists a sequence $\{(\omega^k, \mu_k)\} \subset \Gamma_{\omega^0}$ such that $u_{j_0}^k \to 0$. Since $g_{j_0}(x^k)$ is bounded, we have

$$u_{j_0}^k g_{j_0}(x^k) \to 0, \quad k \to \infty.$$

Moreover, from (8),

$$u_{j_0}^k g_{j_0}(x^k) = \mu_k u_{j_0}^0 g_{j_0}(x^0) \to \bar{\mu} u_{j_0}^0 g_{j_0}(x^0) < 0, \quad k \to \infty.$$

This is a contradiction.

Second, we show that $\bar{\lambda} \notin \partial\mathbb{R}_{++}^p$. In fact, if $\bar{\lambda} \in \partial\mathbb{R}_{++}^p$, then there exists a sequence $\{(\omega^k, \mu^k)\} \subset \Gamma_{\omega^0}$ such that $\lambda_{j_0}^k \to \bar{\lambda}_{j_0} = 0 (k \to \infty)$ for some j_0. Noticing that $\sum_{i=1}^p \lambda_i^0 = 1$ and from (4), it follows that

$$p(1 - \mu_k) + (p\mu_k - p - \mu_k) \sum_{i=1}^p \lambda_i^k + \mu_k = 0.$$

As $k \to \infty$, since $\mu_* \in (0,1)$, the above equation implies that $\sum_{i=1}^p \bar{\lambda}_i = 1$. Take the j_0th equation of (4),

$$(1 - \mu_k)\left(1 - \sum_{i=1}^p \lambda_i^k\right) - \mu_k(\lambda_{j_0}^k - \lambda_{j_0}^0) = 0. \tag{12}$$

As $k\to\infty$ we see that $\lambda^k_{j_0}\to\lambda^0_{j_0}=0$. But $\lambda^0\in\mathbb{R}^p_{++}$, this gives the desired contradiction.

Hence we obtain that there exists a sequence $\{(\omega^k,\mu_k)\}\subset\Gamma_{\omega^0}$ such that $g_j(x^k)\to 0$ for some $j\in I$, and hence $u^k_j g_j(x^k)\to 0$. From (8) and $\bar\mu\in(0,1)$ we observe that $u^k_j g_j(x^k)=\bar\mu u^0_j g_j(x^0)<0$, this is a contradiction.

As a conclusion, (c) is the only possible case. Clearly $(\bar\omega,\bar\mu)\in\bar\Omega\times\mathbb{R}^{p+m}_{+}\times\{0\}$, and $\bar\omega$ is a solution of the KKT system. □

By Theorem 3.4, for almost all $\omega^0\in\Omega\times\mathbb{R}^{p+m}_{++}\times\mathbb{R}^s\times\{0\}$, the homotopy equation (2) generates a smooth curve Γ_{ω^0} which we call the homotopy path, and the ω component of $(\omega(s),\mu(s))$ in the homotopy path, is the solution of (1) as $\mu(s)\to 0$. We can propose following algorithm to track numerically Γ_{ω^0} from $(\omega^0,1)$ until $\mu(s)\to 0$. A simple numerical example will be given later.

Algorithm 3.1 (MOP)'s Euler–Newton method

Step 0: Give an initial point $(\omega^0,1)\in\Omega\times\Lambda^{++}\times\mathbb{R}^m_{++}\times\{0\}\times\{1\}$, an initial steplength $h_0>0$ and three small positive numbers $\epsilon_1,\epsilon_2,\epsilon_3$. Let $k:=0$;

Step 1: Compute the direction η^k of the predictor step:

(a) Compute a unit tangent vector $\xi^k\in\mathbb{R}^{n+p+m+s+1}$ of Γ_{ω^0} at (ω^k,μ_k);

(b) Determine the direction η^k of the predictor step as follows:

$$\textit{If}\ \operatorname{sign}\begin{vmatrix} DH_{\omega^0}(\omega^k,\mu_k)\\ \xi^{kT}\end{vmatrix}=(-1)^{p+m+s+pm+ps+ms+1},\ \textit{take}\ \eta^k=\xi^k;$$

$$\textit{If}\ \operatorname{sign}\begin{vmatrix} DH_{\omega^0}(\omega^k,\mu_k)\\ \xi^{kT}\end{vmatrix}=(-1)^{p+m+s+pm+ps+ms},\ \textit{take}\ \eta^k=-\xi^k.$$

Step 2: Compute a corrector point (ω^{k+1},μ_{k+1}):

$$(\bar\omega^k,\bar\mu_k)=(\omega^k,\mu_k)+h_k\eta^k,$$

$$(\omega^{k+1},\mu_{k+1})=(\bar\omega^k,\bar\mu_k)-DH_{\omega_0}(\bar\omega^k,\bar\mu_k)^+H_{\omega^0}(\bar\omega^k,\bar\mu_k),$$

where $DH_{\omega^0}(\omega,\mu)^+=DH_{\omega^0}(\omega,\mu)^T(DH_{\omega^0}(\omega,\mu)DH_{\omega^0}(\omega,\mu)^T)^{-1}$ is the Moore–Penrose inverse of $DH_{\omega^0}(\omega,\mu)$.

If $\|H_{\omega^0}(\omega^{k+1},\mu_{k+1})\|\leqslant\epsilon_1$, let $h_{k+1}=\min\{h_0,2h_k\}$, and go to Step 3.

If $\|H_{\omega^0}(\omega^{k+1},\mu_{k+1})\|\in(\epsilon_1,\epsilon_2)$, let $h_{k+1}=h_k$, and go to Step 3.

If $\|H_{\omega^0}(\omega^{k+1},\mu_{k+1})\|\geqslant\epsilon_2$, let $h_{k+1}=\max\{\frac{1}{2}h_0,\frac{1}{2}h_k\}$, and go to Step 2.

Step 3: If $\omega^{k+1}\in\bar\Omega\times\mathbb{R}^{p+m}_{+}\times\mathbb{R}^s$ and $\mu_{k+1}>\epsilon_3$, let $k:=k+1$ and go to Step 1. If $\omega^{k+1}\in\bar\Omega\times\mathbb{R}^{p+m}_{+}\times\mathbb{R}^s$ and $\mu_{k+1}<-\epsilon_3$, let $h_k:=h_k\frac{\mu_k}{\mu_k-\mu_{k+1}}$, go to Step 2 and recompute (ω^{k+1},μ_{k+1}) for the initial point (ω^k,μ_k).

If $\omega^{k+1} \notin \bar{\Omega} \times \mathbb{R}_+^{p+m} \times \mathbb{R}^s$, let $h_k := \frac{h_k}{2} \frac{\mu_k}{\mu_k - \mu_{k+1}}$, go to Step 2 and recompute $(\omega^{k+1}, \mu_{k+1})$ for the initial point (ω^k, μ_k).
If $\omega^{k+1} \in \bar{\Omega} \times \mathbb{R}_+^{p+m} \times \mathbb{R}^s$, and $|\mu_{k+1}| \leqslant \epsilon_3$, then stop.

Remark 3.1 A unit tangent vector ξ of Γ_{ω^0} at some point $(\omega, \mu) \in \Gamma_{\omega^0}$ satisfies

$$DH_{\omega^0}(\omega(s), \mu(s))\xi = 0. \tag{13}$$

Since 0 is a regular value of H_{ω^0}, that is, $DH_{\omega^0}(\omega(s), \mu(s))$ is a matrix of full row rank, so ξ as determined by the (13) has two opposite directions. One (the possible direction) makes s increase, and the other makes s decrease. There are many successful methods to numerically implement the problem, so we will not deal with it in detail.

Remark 3.2 In the numerical implementation of the homotopy algorithm, another key problem lies in the choice of the predictor direction. In order to enforce the algorithm smoothly and not to lead us back to the initial point, we must go along the positive direction. Reference [8] tells us that the positive direction η at any point (ω, μ) on Γ_{ω^0} keeps the sign of the determinant

$$\begin{vmatrix} DH_{\omega^0}(\omega, \mu) \\ \eta^T \end{vmatrix}$$

invariant. By the following proposition, we show that the predictor direction of Algorithm 3.1 is effective.

Proposition 3.1 *Let Γ_{ω^0} be a smooth curve of $H_{\omega^0}^{-1}(0)$. Then, the direction η^0 of the predictor step at the initial point $(\omega^0, 1)$ satisfies*

$$\operatorname{sign} \begin{vmatrix} DH_{\omega^0}(\omega^0, 1) \\ \eta^{0T} \end{vmatrix} = (-1)^{p+m+s+pm+ps+ms+1}.$$

Proof By Theorem 3.4, we know that the unit tangent vector ξ^0 of Γ_{ω^0} at $(\omega^0, 1)$ satisfies

$$\xi_1^0 = -M_1^{-1} M_2 \xi_2^0.$$

By a simple computation, we have

$$|M_1| = (-1)^q |\nabla h(x^0) \nabla h(x^0)^T| \prod_{i=1}^m g_i(x^0),$$

where

$$q = p + s + pm + ps + ms.$$

Hence,

$$\begin{vmatrix} DH_{\omega^0}(\omega^0,1) \\ \xi^{0T} \end{vmatrix} = \begin{vmatrix} M_1 & M_2 \\ \xi_1^{0T} & \xi_2^{0T} \end{vmatrix} = \begin{vmatrix} M_1 & M_2 \\ -M_2^T M_1^{-T} \xi_2^0 & \xi_2^0 \end{vmatrix}$$

$$= \begin{vmatrix} M_1 & M_2 \\ 0 & 1 + M_2^T M_1^{-T} M_1^{-1} M_2 \end{vmatrix} \xi_2^0 = |M_1|(1 + M_2^T M_1^{-T} M_1^{-1} M_2)\xi_2^0.$$

By the definition of the direction of the predictor step, $\eta_2^0 < 0$ and

$$\text{sign}|M_1| = \text{sign}\{(-1)^q |\nabla h(x^0)\nabla h(x^0)^T| \prod_{i=1}^m g_i(x^0)\} = (-1)^{p+m+s+pm+ps+ms}$$

so

$$\text{sign}\begin{vmatrix} DH_{\omega^0}(\omega^0,1) \\ \eta^{0T} \end{vmatrix} = (-1)^{p+m+s+pm+ps+ms+1}.$$

□

4 The Homotopy Method for Convex Multiobjective Program

For convex multiobjective programming problem (CMOP), we can replace Condition (A1) by the following condition:

(A4) there exists $z^0 \in \Omega$ such that $\Omega(z^0) := \{x \in \bar{\Omega} : (x - z^0)^T \nabla f_j(x) < 0, j = 1, 2, \ldots, p\}$ is bounded.

Theorem 4.1 *Suppose that all the f_i, g_j are twice continuously differentiable convex functions, and h_k are linear functions. If $\Omega \neq \emptyset$ and Conditions* (A2), (A4) *hold, then problem* (1) *has a solution.*

Proof By the discussion in the previous section, we only need to show that the projection of Γ_{ω^0} onto x component is bounded after proving its λ component is bounded. Suppose that the projection of Γ_{ω^0} onto x component is unbounded. Then there exists a sequence $\{(\omega^k, \mu_k)\} \subset \Gamma_{\omega_0}$ such that $\|x^k\| \to \infty$, $k \to \infty$. From the homotopy equation (2), we have that

$$(1 - \mu_k)(\nabla f(x^k)\lambda^k + \nabla g(x^k)u^k) + \nabla h(x^k)v^k + \mu_k(x^k - x^0) = 0.$$

Thus,

$$(1 - \mu_k)[(x^k - z^0)^T \nabla f(x^k)\lambda^k + (x^k - z^0)^T \nabla g(x^k)u^k] + (x^k - z^0)^T \nabla h(x^k)v^k + \mu_k(x^k - z^0)^T(x^k - x^0) = 0.$$

By the convexity of g_j, h_k, and $z_0 \in \Omega$, we can deduce that

$$\begin{aligned}
&(1-\mu_k)(x^k - z^0)^T \nabla f(x^k)\lambda^k \\
&= -\mu_k (x^k - z^0)^T (x^k - x^0) \\
&\quad - (1-\mu_k)(x^k - z^0)^T \nabla g(x^k)^T u^k - (x^k - z^0)^T \nabla h(x^k) v^k \\
&= -\mu_k \|x^k - x^0\|^2 - \mu_k (x^0 - z^0)^T (x^k - x^0) \\
&\quad + (1-\mu_k)(z^0 - x^k)^T \nabla g(x^k) u^k + (z^0 - x^k)^T \nabla h(x^k) v^k \\
&\le -\frac{1}{2}\mu_k (\|x^k - x^0\|^2 - \|x^0 - z^0\|^2) \\
&\quad + (1-\mu_k)(g(z^0) - g(x^k)) u^k + (h(z^0) - h(x^k)) v^k \\
&\le -\frac{1}{2}\mu_k (\|x^k - x^0\|^2 - \|x^0 - z^0\|^2) - (1-\mu_k) g(x^k) u^k \\
&= -\frac{1}{2}\mu_k (\|x^k - x^0\|^2 - \|x^0 - z^0\|^2) - (1-\mu_k)\mu_k g(x^0) u^0,
\end{aligned}$$

where the last equation holds by using (8). Noticing that $\|x^k\| \to \infty$, letting $k \to \infty$ in the above inequality, we obtain

$$\begin{aligned}
&\mu_k^{-1}(1-\mu_k)(x^k - z^0)^T \nabla f(x^k)\lambda^k \\
&\le -\frac{1}{2}(\|x^k - x^0\|^2 - \|z^0 - x^0\|^2) - (1-\mu_k) g(x^0) u^0 \to -\infty.
\end{aligned}$$

Hence, there exists $j_0 \in \{1, 2, \dots, p\}$, such that $(x^k - z^0)^T \nabla f_{j_0}(x^k) < 0$. This contradicts Condition (A4). Thus the x-component of Γ_{ω_0} is bounded. □

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