Asymptotic analysis of a random walk with a history-dependent step length

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(July 10, 2004)

Abstract

We study an unbiased, discrete-time random walk on the nonnegative integers, with the origin absorbing, and a history-dependent step length. Letting y denote the maximum distance the walker has ever been from the origin, steps that do not change y have length v, while those that increase y (taking the walker to a site that has never been visited), have length n. The process serves as a simplified model of spreading in systems with an infinite number of absorbing configurations. Asymptotic analysis of the probability generating function shows that, for large t, the survival probability decays as $S(t) \sim t^{-\delta}$, with $\delta = v/2n$. Our expression for the decay exponent is in agreement with results obtained via numerical iteration of the transition matrix.

I. INTRODUCTION

Random walks with absorbing and/or reflecting boundaries and/or memory serve as important models in statistical physics, often admitting an exact analysis. Among the many examples are equilibrium models for polymer adsorption [1,2] and absorbing-state phase transitions [3]. Another motivation is provided by the spreading of an epidemic in a medium with a long memory [4]. In this work we discuss a process where the susceptibility changes after the first infection and remains constant thereafter.

In addition to the intrinsic interest of such an infection with memory, our study is motivated by the spread of activity in models exhibiting an infinite number of absorbing configurations, (INAC) typified by the pair contact process [5,6]. Anomalies in critical spreading for INAC, such as continuously variable critical exponents, have been traced to

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a long memory in the dynamics of the order parameter, ρ, due to coupling to an auxiliary field that remains frozen in regions where $\rho = 0$ [6,7]. INAC appears to be particularly relevant to the transition to spatiotemporal chaos, as shown in a recent study of a coupled-map lattice with "laminar" and "turbulent" states, which revealed continuously variable spreading exponents [8]. Grassberger, Chaté and Rousseau [4] proposed that spreading in INAC could be understood by studying a model with a *unique* absorbing configuration, but in which the spreading rate of activity into previously inactive regions is different than for revisiting a region that has already been active.

In light of the anomalies found in spreading in models with INAC or with a memory, we are interested in studying the effect of such a memory on the scaling behavior in a model whose asymptotic behavior can be determined exactly. Of particular interest is the survival probability $S(t)$ (i.e., not to have fallen into the absorbing state up to time t). The simplest example of such a model is an unbiased random walk on the nonnegative integers, with the origin absorbing, for which $S(t) \sim t^{-\delta}$ with $\delta = 1/2$. It was recently shown that such a walker exhibits a continuously variable exponent δ when subject to a mobile, partial reflector. The latter is initially one site to the right of the walker. Each time the walker steps onto the site occupied by the reflector, it is reflected one step to the left with probability r (it remains at its new location with probability $1-r$); in either case, the reflector is pushed forward one site in this encounter. The survival exponent $\delta = (1+r)/2$ in this process [9]. Since the reflector effectively records the *span* of the walk (i.e., the rightmost site yet visited), its interaction with the walker represents a memory.

In the present work, we study a random walk with memory of a different form: if the target site x lies in the region that has been visited before (that is, if x itself has been visited, or lies between two sites that have been visited), then the step length is v; otherwise the step length is n. If $v > n$, the random walk evolves in a *hostile* enviroment, while for $v < n$, the enviroment may be regarded as *friendly*. On the basis of an exact solution for the probability generating function, we obtain the decay exponent δ.

The balance of this paper is organized as follows. In Sec. II we analyze the specific case of a random walk in a hostile enviroment with $v = 2$ and $n = 1$, present the solution of the generating function, and obtain the asymptotic behavior of the survival probability. In Sec. III we extend the analysis to arbitrary step lengths v and n (with v and n natural numbers). In Sec. IV we present exact numerical results for finite times (from iteration of the probability transfer matrix) that complement and extend the asymptotic analysis. Sec. V contains a brief summary and discussion.

II. RANDOM WALK IN A HOSTILE ENVIROMENT

A. Model and Generating Function

Consider an unbiased, discrete time random walk on the nonnegative integers, with the origin absorbing. We denote the position of the walker at time t by x_t and suppose that $x_0 = 1$. To define precisely the history dependence, let $y_t \equiv \max_t\{x_t\}$. Then, if $x_t \leq y_t - 2$, the walker jumps two lattice spacings to the left or the right. If, however, $x_t = y_t$, it can move (with equal probability) to $y_t - 2$, or to $y_t + 1$. (In the latter case $y_{t+1} = y_t + 1$. Notice that $y_t - x_t \geq 0$ must be even.) Let sites 1,...,y_t define the *known region*; steps to sites within the known region have length two, while those that take the walker into the *unknown* region ($x > y_t$) are of unit length.

Evidently, the process x_t is non-Markovian, since the transition probability into a given site depends on whether it lies in the known or the unknown region. We can however transform the model to a Markov process by enlarging the state space [10] to include the boundary between the two regions. Evidently, the stochastic process (x_t, y_t) is Markovian. The transitions (all with probability 1/2) for the Markov chain are restricted to the set $E \subset \mathbb{Z}^2$ specified by

$$E = \{(x,y) \in \mathbb{Z}^2 : x \geq -1, \quad y \geq 1, \quad x \leq y, \quad y - x \text{ is even}\}$$

as represented in Fig. 1.

Let $P(x,y,t)$ denote the probability of state (x,y) (for $x > 0$), at time t. $P(x,y,t)$ follows the evolution equation

$$P(x,y,t+1) = \frac{1}{2}P(x+2,y,t) + \frac{1}{2}P(x-2,y,t), \qquad \text{for } x < y, \tag{1}$$

with $P(1,1,t) = \delta_{0,t}$. Eq. (1) is subject to two boundary conditions. The first is the absorbing condition for $x \leq 0$

$$P(x,y,t) = 0, \qquad \text{for } x \leq 0. \tag{2}$$

The second applies along the diagonal $x = y$. In this case, it is convenient to define $D(y,t) \equiv P(y,y,t)$. On the diagonal the evolution equation is

$$D(y,t+1) = \frac{1}{2}\,D(y-1,t) + \frac{1}{2}P(y-2,y,t), \qquad \text{for } y \geq 2. \tag{3}$$

To solve the problem specified by Eqs. (1) - (3), we introduce a generating function:

$$\hat{P}(x,y,z) = \sum_{t=0}^{\infty} P(x,y,t)\, z^t, \tag{4}$$

where $0 \leq z \leq 1$. Multiplying Eq. (1) by z^t, summing over t and shifting the sum index where necessary, one finds that the generating function satisfies

$$\frac{1}{z}\hat{P}(x,y) = \frac{1}{2}\hat{P}(x+2,y) + \frac{1}{2}\hat{P}(x-2,y), \qquad \text{for } x \leq y-4 \tag{5}$$

$$\frac{1}{z}\hat{P}(y-2,y) = \frac{1}{2}\hat{D}(y) + \frac{1}{2}\hat{P}(y-4,y), \qquad \text{for } x = y-2, \tag{6}$$

(we drop the argument z for brevity), where $\hat{D}(y)$ is defined by an expression analogous to Eq. (4). The initial condition implies $\hat{D}(1) = 1$; the boundary conditions are

$$\hat{P}(x,y) = 0, \qquad \text{for } x \leq 0, \tag{7}$$

$$\frac{1}{z}\hat{D}(y) = \frac{1}{2}\hat{D}(y-1) + \frac{1}{2}\hat{P}(y-2,y), \qquad \text{for } y \geq 2. \tag{8}$$

Next, we focus on Eq. (8) in order to eliminate $\hat{D}(y)$ in Eq. (6), and then find a recurrence relation for $\hat{P}(x,y)$.

Substituting Eq. (6) in Eq. (8) we have

$$\left[1-\left(\frac{z}{2}\right)^2\right]\ \hat{D}(y)=\frac{z}{2}\ \hat{D}(y-1)+\left(\frac{z}{2}\right)^2\ \hat{P}(y-4,y). \tag{9}$$

If we subtract $z/2$ times Eq. (9), evaluated at $y-1$, from the corresponding equation for y, we find

$$\begin{aligned}\left[1-\left(\frac{z}{2}\right)^2\right]\ \left[\hat{D}(y)-\frac{z}{2}\ \hat{D}(y-1)\right]\ &=\frac{z}{2}\left[\hat{D}(y-1)-\frac{z}{2}\ \hat{D}(y-2)\right]+\\ &+\left(\frac{z}{2}\right)^2\left[\hat{P}(y-4,y)-\frac{z}{2}\hat{P}(y-5,y-1)\right].\end{aligned}$$

Using Eq. (8) we eliminate $\hat{D}(y)$, $\hat{D}(y-1)$ and $\hat{D}(y-2)$ to obtain

$$\begin{aligned}\frac{z}{2}\left[1-\left(\frac{z}{2}\right)^2\right]\ \hat{P}(y-2,y)=\left(\frac{z}{2}\right)^2\hat{P}(y-3,y-1)+&\\ +\left(\frac{z}{2}\right)^2\left[\hat{P}(y-4,y)-\frac{z}{2}\hat{P}(y-5,y-1)\right]&,\end{aligned}$$

yielding the recurrence relation

$$\left[\frac{4-z^2}{2z}\right]\ \hat{P}(y-2,y)-\hat{P}(y-4,y)=\hat{P}(y-3,y-1)-\frac{z}{2}\hat{P}(y-5,y-1). \tag{10}$$

Eq. (5) relates $\hat{P}(x,y)$ for different x, at fixed y. We therefore impose separation of variables and write, for $x\le y-2$,

$$\hat{P}(x,y)=\hat{A}(x)\ \hat{B}(y). \tag{11}$$

Eq. (2) requires $\hat{A}(x)=0$ for $x\le 0$, which is satisfied if

$$\hat{A}(x)=\begin{cases}\lambda^x-\lambda^{-x} & \text{, for } x\ge 0,\\ 0 & \text{, for } x<0.\end{cases} \tag{12}$$

In this context, Eq. (5) implies:

$$\lambda=\left(\frac{1}{z}+\sqrt{\frac{1}{z^2}-1}\right)^{1/2}. \tag{13}$$

(Note that use of the second solution, $\lambda_-=(z^{-1}-\sqrt{z^{-2}-1})^{1/2}=\lambda^{-1}$, would simply result in a change in the sign of $\hat{A}$.) Substituting Eq. (11) in Eq. (10) we find

$$\frac{\hat{B}(y)}{\hat{B}(y-1)}=\frac{2z\ \hat{A}(y-3)-z^2\ \hat{A}(y-5)}{(4-z^2)\ \hat{A}(y-2)-2z\ \hat{A}(y-4)}. \tag{14}$$

B. The survival probability

The survival probability is

$$S(t) = \sum_{y=2}^{\infty} \sum_{k=0}^{\left[\frac{y-1}{2}\right]} P(y-2k, y, t),$$

where [ ] denotes the integer part of its argument. The corresponding generating function is:

$$\begin{aligned} \hat{S}(z) &= \sum_{t=0}^{\infty} S(t)\; z^t \\ &= \hat{S}_P(z) + \hat{S}_D(z), \end{aligned} \tag{15}$$

where $\hat{S}_P(z) = \sum_{y=2}^{\infty} \sum_{k=1}^{[(y-1)/2]} \hat{P}(y-2k, y)$ and $\hat{S}_D(z) = \sum_{y=2}^{\infty} \hat{D}(y)$. We study these series separately. To begin, we insert Eq.(11) in $\hat{S}_P$ to obtain

$$\hat{S}_P(z) = \sum_{y=2}^{\infty} \sum_{k=1}^{\left[\frac{y-1}{2}\right]} \hat{A}(y-2k)\; \hat{B}(y). \tag{16}$$

Next we examine $\hat{S}_D(z)$. Iterating Eq. (8), we have

$$\hat{D}(y) = \left(\frac{z}{2}\right)^{y-1} + \sum_{j=2}^{y} \left(\frac{z}{2}\right)^{y+1-j} \hat{P}(j-2, j). \tag{17}$$

Summing Eq. (17) over $y \geq 2$ we find

$$\hat{S}_D(z) = \frac{z}{2-z} \left[1 + \sum_{y=2}^{\infty} \hat{A}(y-2)\; \hat{B}(y)\right]. \tag{18}$$

C. Asymptotic analysis

We address the *Tauberian* problem [11] of extracting the large-t asymptotics of $S(t)$ from the dominant singularity of its generating function $\hat{S}(z)$, as $z \uparrow 1$. In order to study this limit, let $z = 1 - \epsilon$, with $\epsilon \downarrow 0$. We will show that as $t \to \infty$, the dominant contribution to the survival probability comes from $\hat{S}_P(z)$.

To determine the asymptotic behavior of $\hat{S}_P$, we analyze $\hat{B}(y)$ and the sum $\sum_{k=1}^{[(y-1)/2]} \hat{A}(y-2k)$ separately. First, we focus on $\hat{B}$; in light of Eq. (14), it is convenient to write,

$$\hat{B}(y) = \hat{B}(3) \prod_{k=4}^{y} \frac{\hat{B}(k)}{\hat{B}(k-1)}. \tag{19}$$

¿From Eq. (11) we have $\hat{B}(3) = \hat{P}(1,3)\ /\ \hat{A}(1)$, with $\hat{A}(1) \simeq \sqrt{2\epsilon}$, as $\epsilon \downarrow 0$. (We use the symbol "$\simeq$" to indicate asymptotic equality as $\epsilon \downarrow 0$.) On the other hand, Eq. (8) implies that $\hat{P}(1,3) = \frac{2}{z}\ \hat{D}(3) - \hat{D}(2)$, where $\hat{D}(2) = z/2$. Iterating Eq. (3) we have

$$D(3,t) = \begin{cases} \frac{1}{2^t} & \text{, if } t \text{ is odd} \\ 0 & \text{, if } t \text{ is even} \end{cases}$$

so that $\hat{D}(3) = 2z\ /\ (4 - z^2)$. Evidently, $\hat{P}(1,3) \simeq \frac{1}{6}$, as $\epsilon \downarrow 0$, and therefore

$$\hat{B}(3) \simeq \frac{1}{6\sqrt{2\epsilon}}. \tag{20}$$

The ratio $\hat{B}(k)/\hat{B}(k-1)$ may be analyzed by inserting Eq. (12) in Eq. (14)

$$\frac{\hat{B}(k)}{\hat{B}(k-1)} = \frac{\lambda^k(2z\lambda^{-3} - z^2\lambda^{-5}) - \lambda^{-k}(2z\lambda^3 - z^2\lambda^5)}{\lambda^k[(4-z^2)\lambda^{-2} - 2z\lambda^{-4}] - \lambda^{-k}[(4-z^2)\lambda^2 - 2z\lambda^4]}$$

For small ϵ, $\lambda = 1 + \frac{1}{2}\sqrt{2\epsilon} + \mathcal{O}(\epsilon)$. Then,

$$\frac{\hat{B}(k)}{\hat{B}(k-1)} \simeq \frac{\lambda^k - \lambda^{-k} - \frac{1}{2}\sqrt{2\epsilon}(\lambda^k + \lambda^{-k})}{\lambda^k - \lambda^{-k} + \sqrt{2\epsilon}(\lambda^k + \lambda^{-k})}.$$

Now, letting $\Lambda = \ln(\lambda)$, we have

$$\frac{\hat{B}(k)}{\hat{B}(k-1)} \simeq \frac{\tanh\left(\frac{\sqrt{2\epsilon}}{2}\,k\right) - \frac{1}{2}\sqrt{2\epsilon}}{\tanh\left(\frac{\sqrt{2\epsilon}}{2}\,k\right) + \sqrt{2\epsilon}}.$$

Letting $\phi_k = \tanh\left(\frac{\sqrt{2\epsilon}}{2}k\right)$, this yields,

$$\sum_{k=4}^{y} \ln\left[\frac{\hat{B}(k)}{\hat{B}(k-1)}\right] \simeq -\left(\frac{3}{2}\right)\sqrt{2\epsilon}\sum_{k=4}^{y}\frac{1}{\phi_k}.$$

Approximating the sum by an integral, we have

$$\begin{aligned}\sum_{k=4}^{y}\frac{1}{\phi_k} &\simeq \frac{2}{\sqrt{2\epsilon}}\int_{2\sqrt{2\epsilon}}^{\frac{\sqrt{2\epsilon}}{2}y}\frac{1}{\tanh(w)}\,dw \\ &\simeq \frac{2}{\sqrt{2\epsilon}}\ln\left[\frac{\sinh\left(\frac{\sqrt{2\epsilon}}{2}y\right)}{\sinh\left(2\sqrt{2\epsilon}\right)}\right],\end{aligned}$$

and hence,

$$\prod_{k=4}^{y}\frac{\hat{B}(k)}{\hat{B}(k-1)} \simeq C\,\frac{\epsilon^{\frac{3}{2}}}{\sinh^3\left[\frac{\sqrt{2\epsilon}}{2}y\right]}, \tag{21}$$

where C is a constant. Inserting Eqs. (20) and (21) in Eq. (19) we find,

$$\hat{B}(y) \sim \frac{\epsilon}{\sinh^3\left(\frac{\sqrt{2\epsilon}}{2}y\right)}. \tag{22}$$

(By "$\sim$" we mean asymptotic proportionality as $\epsilon \downarrow 0$, i.e., multiplicative constants are ignored.) Finally, we note that as $\epsilon \downarrow 0$,

$$\sum_{k=1}^{\left[\frac{y-1}{2}\right]} \hat{A}(y-2k) \simeq \frac{\left[\hat{A}\left(\frac{y}{2}\right)\right]^2}{2\sqrt{2\epsilon}} \sim \epsilon^{-\frac{1}{2}} \sinh^2\left(\frac{\sqrt{2\epsilon}}{4}y\right). \tag{23}$$

With these results, we are in a position to analyze the asymptotic behavior of $\hat{S}_P$ and $\hat{S}_D$.

First, we determine the asymptotic behavior of $\hat{S}_P(z)$. Substituting Eqs. (23) and (22) in Eq. (16) we find

$$\hat{S}_P \sim \epsilon^{\frac{1}{2}} \sum_{y=2}^{\infty} \frac{\sinh^2\left(\frac{\sqrt{2\epsilon}}{4}y\right)}{\sinh^3\left(\frac{\sqrt{2\epsilon}}{2}y\right)}.$$

Denoting the sum by $H(\epsilon)$, we have

$$H(\epsilon) \sim \epsilon^{-1/2} \int_{\sqrt{2\epsilon}}^{\infty} \frac{\sinh^2\left(\frac{1}{2}w\right)}{\sinh^3(w)}\, dw.$$

Let us denote the integrand by $f(w)$ and the integral by $I(\epsilon)$. Since $f(w)$ has a pole of order 1, we introduce the Laurent expansion

$$f(w) = \frac{1}{4}\, w^{-1} + \sum_{k=0}^{\infty} a_k\, w^k$$

and integrate the series term by term. Noting that the dominant contribution, as $\epsilon \downarrow 0$, comes from the first term, we have

$$I(\epsilon) \simeq \frac{1}{4} \int_{\sqrt{2\epsilon}}^{1} w^{-1}\, dw \simeq -\frac{1}{8}\, \ln(\epsilon).$$

Thus,

$$\hat{S}_P \sim -\ln(\epsilon).$$

Using the same line of reasoning, it can be shown $\lim_{\epsilon\downarrow 0} \hat{S}_D/\hat{S}_P = 0$. Therefore, the dominant singular behavior of $\hat{S}$ as $z \uparrow 1$ is given by:

$$\hat{S}(z) \sim -\ln(1-z).$$

The coefficient of z^t in the expansion of $-\ln(1-z)$ is t^{-1}, and so the survival probability decays asymptotically as t^{-1}.

III. ARBITRARY STEP LENGTHS

In this section, we generalize the analysis of Sec. II to a walker with an arbitrary history-dependent step length. Let v be the step length for target sites in the known region, and n the step length in case the target site lies in the unknown region. We consider the Markov chain (x_t, y_t), with y_t as defined in Sec. II and transitions with probability 1/2. The probability $P(x, y, t)$ follows the equation

$$P(x, y, t+1) = \frac{1}{2}P(x+v, y, t) + \frac{1}{2}P(x-v, y, t), \qquad \text{for } x < y, \tag{24}$$

with $P(1, 1, t) = \delta_{t,0}$. Eq. (24) is subject to two boundary conditions, the first, Eq. (2), is due to the absorbing condition. The second applies along the diagonal $x = y$. Defining $D(y, t) = P(y, y, t)$, as before, we have

$$D(y, t+1) = \frac{1}{2}\, D(y-n, t) + \frac{1}{2}P(y-v, y, t), \qquad \text{for } y \geq n+1. \tag{25}$$

Introducing the generating functions $\hat{P}(x, y)$ and $\hat{D}(y)$ as in Sec. II, one readily finds,

$$\frac{1}{z}\hat{P}(x, y) = \frac{1}{2}\hat{P}(x+v, y) + \frac{1}{2}\hat{P}(x-v, y), \quad \text{for } x \leq y-2v, \tag{26}$$

$$\frac{1}{z}\hat{P}(y-v, y) = \frac{1}{2}\hat{D}(y) + \frac{1}{2}\hat{P}(y-2v, y), \qquad \text{for } x = y-v. \tag{27}$$

The initial condition is $\hat{D}(1) = 1$, and the boundary conditions are

$$\hat{P}(x, y) = 0, \qquad \text{for } x \leq 0 \tag{28}$$

$$\frac{1}{z}\hat{D}(y) = \frac{1}{2}\hat{D}(y-n) + \frac{1}{2}\hat{P}(y-v, y), \quad \text{for } y \geq n+1. \tag{29}$$

Proceeding as in Sec. II, one finds the recurrence relation:

$$\left[\frac{4-z^2}{2z}\right] \hat{P}(y-v, y) - \hat{P}(y-2v, y) = \hat{P}(y-v-n, y-n) - \frac{z}{2}\hat{P}(y-2v-n, y-n). \tag{30}$$

The solution for $\hat{P}(x, y)$ is again of the form of Eq. (11), with $\hat{A}$ again given by Eq. (12), but with,

$$\lambda = \left(\frac{1}{z} + \sqrt{\frac{1}{z^2} - 1}\right)^{1/v}$$

With this, one readily finds,

$$\frac{\hat{B}(y)}{\hat{B}(y-n)} = \frac{2z\, \hat{A}(y-v-n) - z^2\, \hat{A}(y-2v-n)}{(4-z^2)\, \hat{A}(y-v) - 2z\, \hat{A}(y-2v)}. \tag{31}$$

Since $y = nj + 1$ and $x = y - vk = nj - vk + 1$, $\hat{S}(z)$ is given by

$$\hat{S}(z) = \sum_{j=1}^{\infty} \sum_{k=0}^{L} \hat{P}(nj - vk + 1,\ nj + 1), \tag{32}$$

where $L = \left[\frac{nj}{v}\right]$. We define

$$\begin{aligned} \hat{S}_P(z) &= \sum_{j=1}^{\infty} \sum_{k=1}^{L} \hat{P}(nj - vk + 1,\ nj + 1) \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{L} \hat{A}(nj - vk + 1)\ \hat{B}(nj + 1) \end{aligned} \tag{33}$$

and

$$\hat{S}_D(z) = \sum_{j=1}^{\infty} \hat{D}(nj + 1).$$

Iterating Eq. (29) we find

$$\hat{D}(nj + 1) = \left(\frac{z}{2}\right)^{j} + \sum_{k=1}^{j} \left(\frac{z}{2}\right)^{j+1-k} \hat{P}(nk - v + 1, nk + 1). \tag{34}$$

Summing Eq. (34) over $y \geq n+1$, and inserting the expressions found previously for $\hat{P}(x, y)$, we have

$$\hat{S}_D(z) = \frac{z}{2 - z} \left[1 + \sum_{j=1}^{\infty} \hat{A}(nj - v + 1)\ \hat{B}(nj + 1)\right]. \tag{35}$$

In order to determine the asymptotic behavior of Eqs. (33) and (35), we analyze $\hat{A}$, $\hat{B}$ and the sum $\sum_{k=1}^{L} \hat{A}(nj - vk + 1)$ separately. First, we write $\hat{B}$ in the form:

$$\hat{B}(nj + 1) = \hat{B}(nk_0 + 1) \prod_{k=k_0+1}^{j} \frac{\hat{B}(nk + 1)}{\hat{B}(nk - n + 1)}, \tag{36}$$

where k_0 is the smallest positive integer such that the argument of $\hat{A}(x)$ is positive. Note that

$$\hat{B}(nk_0 + 1) \simeq C_1(v, n, k_0)\ \epsilon^{-\frac{1}{2}}, \tag{37}$$

with $C_1(v, n, k_0)$ a coefficient which depends on v, n and k_0. The ratio $\hat{B}(y)/\hat{B}(y - n)$ may be written,

$$\frac{\hat{B}(y)}{\hat{B}(y - n)} = \frac{\lambda^{y}(2z\lambda^{-v-n} - z^2\lambda^{-2v-n}) - \lambda^{-y}(2z\lambda^{v+n} - z^2\lambda^{2v+n})}{\lambda^{y}[(4 - z^2)\lambda^{-v} - 2z\lambda^{-2v}] - \lambda^{-y}[(4 - z^2)\lambda^{v} - 2z\lambda^{2v}]}.$$

For small ϵ, $\lambda \simeq 1 + \frac{1}{v}\sqrt{2\epsilon}$, and

$$\frac{\hat{B}(y)}{\hat{B}(y-n)} \simeq \frac{\lambda^y - \lambda^{-y} - \frac{n}{v}\sqrt{2\epsilon}(\lambda^y + \lambda^{-y})}{\lambda^y - \lambda^{-y} + \sqrt{2\epsilon}(\lambda^y + \lambda^{-y})}.$$

As before, let $\Lambda = \ln(\lambda)$. Then,

$$\frac{\hat{B}(y)}{\hat{B}(y-n)} \simeq \frac{\tanh\left(\frac{\sqrt{2\epsilon}}{v}\, y\right) - \frac{n}{v}\sqrt{2\epsilon}}{\tanh\left(\frac{\sqrt{2\epsilon}}{v}\, y\right) + \sqrt{2\epsilon}}.$$

A calculation analogous to that leading to Eq. (22) then yields,

$$\hat{B}(nj+1) \simeq C_2(v,n,k_0)\, \frac{\epsilon^{\frac{v}{2n}}}{\sinh^{1+\frac{v}{n}}\left[\frac{\sqrt{2\epsilon}}{v}(1+nj)\right]}, \tag{38}$$

with $C_2(v,n,k_0)$ a coefficient which depends on v, n and k_0. Finally, we note that as $\epsilon \downarrow 0$,

$$\sum_{k=1}^{L} \hat{A}(nj - vk + 1) \simeq \frac{\left[\hat{A}\left(\frac{nj+1}{2}\right)\right]^2}{v\sqrt{2\epsilon}} \sim \epsilon^{-\frac{1}{2}} \sinh^2\left[\frac{\sqrt{2\epsilon}}{2v}(nj+1)\right]. \tag{39}$$

With these results, we are in a position to analyze the asymptotic behavior of $\hat{S}_P$ and $\hat{S}_D$.

First, we determine the asymptotic behavior of $\hat{S}_P(z)$. Substituting Eqs. (39) and (38) in Eq. (33) we find

$$\hat{S}_P \sim \epsilon^{\frac{v}{2n}-\frac{1}{2}} \sum_{j=1}^{\infty} \frac{\sinh^2\left[\frac{\sqrt{2\epsilon}}{2v}(nj+1)\right]}{\sinh^{1+\frac{v}{n}}\left[\frac{\sqrt{2\epsilon}}{v}(nj+1)\right]}.$$

As before, we approximate the sum, denoted $H(\epsilon)$, by an integral,

$$H(\epsilon) \sim \frac{v}{n\sqrt{2}}\, \epsilon^{-1/2} \int_{q\epsilon^{1/2}}^{\infty} \frac{\sinh^2\left(\frac{1}{2}w\right)}{\sinh^{1+\frac{v}{n}}(w)}\, dw,$$

with $q = \frac{(1+n)\sqrt{2}}{v}$. Let us denote the integrand by $f(w)$ and the integral by $I(\epsilon)$. First, note that if $v < n$, then $f(w)$ is bounded, and $I(\epsilon)$ converges. Therefore,

$$\hat{S}_P \sim \epsilon^{\frac{v}{2n}-1}, \qquad \text{for } v < n.$$

On the other hand, if $v > n$, then $f(w)$ diverges as $w \downarrow 0$ and decays exponentially for $w \gg 1$. In particular, if $\frac{v}{n} = 2m$, with $m = 1, 2, 3, \ldots$, then $f(w)$ has a pole of order m. Introducing the Laurent expansion

$$f(w) = \sum_{k=0}^{\infty} a_k(m)\, w^k + \sum_{k=1}^{m} b_k(m)\, w^{-k}$$

and integrating the series term by term, we note that the dominant contribution to $\hat{S}_P$, as $\epsilon \downarrow 0$, comes from the term proportional to w^{-1}. Thus,

$$I(\epsilon) \sim b_1(m) \int_{q\epsilon^{1/2}}^{1} w^{-1}\, dw \sim -b_1(m)\ \ln(\epsilon).$$

Since $b_1(m)$ is the residue of $f(w)$ at $w = 0$, we may relate it to an integral around a closed contour containing the origin in the complex-w plane. In this manner we can establish the recurrence relation:

$$b_1(m) = -\frac{1}{2m}\left[b_1(m-1) - b_1(m-2) + b_1(m-3) - \ldots + (-1)^m\, 3b_1(1)\right],$$

where $b_1(1) = 1/4$, as found in Sec. II. Observe that the $b_1(m)$ alternate in sign. Thus,

$$\hat{S}_P \sim (-1)^{\frac{v}{2n}}\ \epsilon^{\frac{v}{2n}-1}\ \ln(\epsilon), \qquad \text{for } v = 2m\,n.$$

Using the same line of reasoning, it can be shown that if $n < v \neq 2m\,n$, then $\hat{S}_P \sim \epsilon^{\frac{v}{2n}-1}$. Moreover, $\lim_{\epsilon\downarrow 0} \hat{S}_D/\hat{S}_P = 0$. Therefore, the dominant singular behavior of $\hat{S}$ as $z \uparrow 1$ is given by:

$$\hat{S}(z) \sim \begin{cases} (-1)^{\frac{v}{2n}}\ (1-z)^{\frac{v}{2n}-1}\ \ln(1-z)\ , & \text{for } v = 2n, 4n, 6n, \ldots, \\ (1-z)^{\frac{v}{2n}-1}\ , & \text{otherwise.} \end{cases}$$

The coefficient of z^t (for large t) in the expansion of $(-1)^{\frac{v}{2n}}(1-z)^{\frac{v}{2n}-1}\ \ln(1-z)$ is proportional to $t^{-\frac{v}{2n}}$ and therefore the survival probability decays asymptotically as $t^{-\frac{v}{2n}}$, for $v = 2mn$. On the other hand, since the coefficient of z^t (for large t) in the expansion of $(1-z)^\alpha$ is proportional to $t^{-\delta}$, with $\delta = 1 + \alpha$, we conclude that the survival probability decays asymptotically as $t^{-v/2n}$, for $v \neq 2m\,n$ as well. Thus, we have $S(t) \sim t^{-\frac{v}{2n}}$ for arbitrary step lengths v and n, which is the result we set out to prove.

IV. NUMERICAL RESULTS

In this section we report exact numerical results for finite times ($t \leq 10^4$) from iteration of the discrete time evolution equations. Consider first the hostile enviroment. Iteration of Eq. (1), subject to the boundary conditions of Eqs. (2) and (3), yields the survival probability, $S(t)$, as shown in Fig. 2. Evidently, $S(t)$ approaches the asymptotic value, $2/t$, at long times. It is interesting to examine the mode of approach to this scaling limit; assuming a power-law correction to scaling term, we write

$$S(t) \simeq \frac{2}{t}\left(1 + \frac{A}{t^\phi}\right), \tag{40}$$

so that the dominant correction to scaling $\sim t^{-(1+\phi)}$. If this form is correct, then at long times $\ln\ln[tS(t)/2] \sim C - \phi \ln t$, where C is a constant. Our results confirm the assumed correction to scaling and yield an exponent of ϕ=1 (see Fig. 2, inset).

We have also analyzed, via iteration, the step-length combinations listed in Table I.

In all cases, the predicted value of δ is confirmed, and the correction to scaling exponent ϕ is unity. We have also verified numerically that, in all the cases studied, the mean position, conditioned on survival $\langle x\rangle_s \sim t^{1/2}$ and that $\langle x^2\rangle_s \sim t$, as is to be expected.

V. DISCUSSION

We have studied the asymptotic survival probability of a random walker on the one-dimensional lattice, with the origin absorbing, and with a step-length that depends on whether the target site lies within the region that has been visited before. In all cases studied, we find that the survival probability decays asymptotically as a power law, $S(t) \sim t^{-\delta}$, where $\delta = v/2n$. Our expression for the decay exponent is in agreement with results obtained via numerical iteration of the transition matrix.

ACKNOWLEDGMENTS

We thank Miguel A. Muñoz for helpful comments, in particular for suggesting the study of the friendly enviroment. We also thank Deepak Dhar for helpful discussions. R.D. and F.F.A. acknowledge financial support from CNPq (Brazil); D. b-A. acknowledges support of NSF (USA) under grant PHY-0140094.

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TABLES

v	n	δ
1	2	1/4
2	3	1/3
3	2	3/4
3	1	3/2
4	1	2

TABLE I. Step lengths studied via iteration

Figure Captions

Fig. 1. Random walk in a hostile enviroment: transitions in the x-y plane.

Fig. 2. Main graph: decay of survival probability in the hostile model with $v=2$ and $n=1$; the equation of the solid line is $S = 2/t$. Inset: $\Delta = \ln[\ln(2/t) - \ln S(t)]$ versus $\ln t$; the slope of the straight line is $\phi = -1$.