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QUANTUM PERFECT STATE TRANSFER ON WEIGHTED JOIN GRAPHS

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This paper studies quantum perfect state transfer on weighted graphs. We prove that the join of a weighted two-vertex graph with any regular graph has perfect state transfer. This generalizes a result of Casaccino *et al.*[1] where the regular graph is a complete graph with or without a missing edge. In contrast, we prove that the half-join of a weighted two-vertex graph with any weighted

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regular graph has no perfect state transfer. As a corollary, unlike for complete graphs, adding weights in complete bipartite graphs does not produce perfect state transfer. We also observe that any Hamming graph has perfect state transfer between each pair of its vertices. The result is a corollary of a closure property on weighted Cartesian products of perfect state transfer graphs. Moreover, on a hypercube, we show that perfect state transfer occurs between uniform superpositions on pairs of arbitrary subcubes, thus generalizing results of Bernasconi *et al.*[2] and Moore and Russell.[3]

Keywords: Perfect state transfer; quantum networks; weighted graphs; join.

1. Introduction

Recently, the notion of perfect state transfer on quantum networks modeled by graphs has received considerable attention in quantum information.[1,2,4–8] A main goal in this line of research is to find and characterize graph structures, which exhibit perfect state transfer between pairs of vertices in the graph. This is a useful property of quantum networks since it facilitates information transfer between locations.

We may conveniently view the problem of perfect state transfer in the context of quantum walks on graphs.[9,10] In this setting, the initial state of the quantum system is described by a unit vector on some initial vertex a. To achieve perfect transfer to a target vertex b at time t, the quantum walk amplitude of the system at time t on vertex b must be of unit magnitude. In other words, we require that $|\langle b|e^{-itA_G}|a\rangle| = 1$, where A_G is the adjacency matrix of the underlying graph G which describes the quantum network.

Christandl *et al.*[4] observed that the Cartesian products of paths of length 3 (two-link hypercubes) admit perfect state transfer between antipodal vertices. They also noted that paths of length 4 or larger do not possess perfect state transfer unless their edges are weighted in a specific manner.[5] In fact, this weighting scheme corresponds closely to the hypercube structure. Feder[11] has extended this observation to a more sophisticated construction in the context of bosons. This crucially shows that edge weights can be useful for achieving perfect state transfer on graphs which are known not to possess the property.

It is known that complete graphs do not have perfect state transfer. But, surprisingly, Casaccino *et al.*[1] observed that adding weighted self-loops on two vertices in a complete graph helps create perfect state transfer between the two vertices. We generalize their observation by considering the join of a weighted two-vertex graph with an arbitrary regular graph. We prove that adding weights also helps for perfect state transfer in this more general case. On the other hand, we show that the half-join between a weighted two-vertex graph with a weighted self-join of an arbitrary regular graph, where each vertex of the two-vertex graph is connected to exactly half of the join graph, has no perfect state transfer for any set of weights. This implies that weights provably do not help in achieving perfect state transfer in a complete bipartite graph. The full connection that is available in the standard join seems crucial in achieving perfect state transfer.

Bernasconi *et al.*[2] gave a complete characterization of perfect state transfer on the hypercubes. In fact, their result holds for any $\mathbb{Z}_2^n$-circulant graph. They proved that perfect state transfer is possible at time $t = \pi/2$ between any pair of vertices. We will refer to this stronger property as universal perfect state transfer. Previously known results on perfect state transfer on other graphs, such as integral circulants[8] and two-link hypercubes $P_3^{\oplus n}$,[4] allow only perfect state transfer between antipodal vertices (which are vertices at the maximum distance from each other). Recent results on integral circulants and other graphs[12] have exhibited perfect state transfer between non-antipodal vertices, but most of these graphs still lack the property of universal perfect state transfer.

We show that weights are useful for universal perfect state transfer on the family of Hamming graphs $K_q^{\oplus n}$, which is a generalization of the hypercube family. We prove this result by extending the observation of Christandl *et al.*[4] to perfect state transfer on weighted Cartesian products. For the hypercube Q_n, we prove a stronger property of universal perfect state transfer. We show that perfect state transfer occurs between uniform superpositions over two arbitrary subcubes of Q_n. This generalizes the results of Bernasconi *et al.*[2] (mentioned above) and also of Moore and Russell[3] on the uniform mixing of a quantum walk on the n cube.

Note that if we allow zero edge weights, then universal perfect state transfer becomes trivial. Assuming that the two source and target vertices are connected, find a path connecting them, assign the hypercubic weights to the edges on this path (as in Ref. 4) and zero weights to the other edges. This shows that universal perfect state transfer can be achieved if zero edge weights are allowed.

2. Preliminaries

For a logical statement $\mathcal{S}$, the Iversonian notation $[[\mathcal{S}]]$ is 1 if $\mathcal{S}$ is true and 0 otherwise.[13] As is standard, we use I_n and J_n to denote the $n \times n$ identity and all$-$one matrices, respectively; we drop the subscript n whenever the context is clear.

The graphs $G = (V, E)$ which we study are finite, mostly simple, undirected, and connected. The adjacency matrix A_G of a graph G is defined as $A_G[u, v] = [[(u, v) \in E]]$. A graph G is called k-regular if each vertex has k adjacent neighbors. That is to say, the neighbor set $\{v \in V : (u, v) \in E\}$ of u has cardinality k for each vertex $u \in V$. In most cases, we also require G to be vertex-transitive, i.e. for any $a, b \in V$ there is an automorphism $\pi \in \mathrm{Aut}(G)$ with $\pi(a) = b$.

In this paper, we also consider edge-weighted graphs $\tilde{G} = (V, E, w)$, where $w : E \to \mathbb{R}$ is a function that assigns weights to edges. In the simplest case, we take an unweighted graph $G = (V, E)$ and add self-loops with weight α to all vertices and assign weight β to all edges; we denote such a graph by $\tilde{G}(\alpha, \beta)$. Note that the adjacency matrix of $\tilde{G}$ is given by $\alpha I + \beta A_G$. Unless otherwise stated, most of our weighted graphs will be of this form.

We denote the complete graph on n vertices by K_n. The Cartesian product $G \oplus H$ of graphs G and H is a graph whose adjacency matrix is $I \otimes A_H + A_G \otimes I$.[14] The binary n-dimensional hypercube Q_n may be defined recursively as $Q_n = K_2 \oplus Q_{n-1}$, for $n \geq 2$, and $Q_1 = K_2$. Similarly, the Hamming graph $H(q, n)$ is defined as $K_q^{\oplus n}$; this may be viewed as a q-ary n-dimensional hypercube.

The *join* $G + H$ of graphs G and H is defined as $\overline{G + H} = \overline{G} \cup \overline{H}$, i.e. we take a copy of G and a copy of H and connect all vertices of G with all vertices of H.[15] We will also consider the weighted join $G +_\rho H$, where we assign a weight of ρ to the edges that connect G and H; more specifically, the adjacency matrix of $G +_\rho H$ is given by

$$\begin{bmatrix} A_G & \rho J \\ \rho J & A_H \end{bmatrix}, \tag{1}$$

with the appropriate dimensions on the two all−one J matrices. A *cone* on a graph G is the graph $K_1 + G$. Similarly, a connected *double* cone on a graph G is the graph $K_2 + G$; similarly, a disconnected double cone is the graph $\overline{K}_2 + G$. When G is the empty graph, the connected double cone is simply the complete graph, whereas the disconnected double cone is the complete graph with a missing edge.[7,1] On the other hand, a connected (or disconnected) double *half-cone* on a graph G is formed by taking K_2 (or $\overline{K}_2$) and $G + G$ and connecting each vertex of the two-vertex graph to exactly one copy of G in the join $G + G$. When G is the empty graph, the double half-cone simply yields a complete bipartite graph. For more background on algebraic graph theory, see the monograph by Biggs.[16]

For a graph $G = (V, E)$, let $|\psi(t)\rangle \in \mathbb{C}^{|V|}$ be a time-dependent amplitude vector over V. The continuous time quantum walk on G is defined using Schrödinger's equation as

$$|\psi(t)\rangle = e^{-itA_G}|\psi(0)\rangle, \tag{2}$$

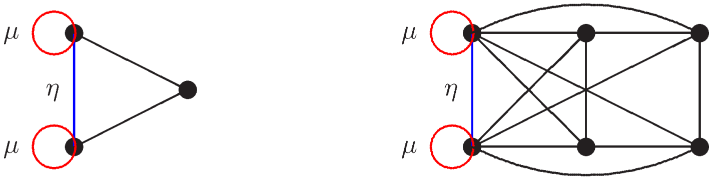
where $|\psi(0)\rangle$ is the initial amplitude vector.[9] Further background on quantum walks on graphs can be found in the survey by Kendon.[10] We say that G has *perfect state transfer* from vertex a to vertex b at time $t^\star$ if

$$|\langle b|e^{-it^\star A_G}|a\rangle| = 1, \tag{3}$$

where $|a\rangle$ and $|b\rangle$ denote the unit vectors corresponding to the vertices a and b, respectively. The graph G has perfect state transfer if there exist distinct vertices a and b in G and time $t^\star \in \mathbb{R}^+$ so that (3) is true. We say that G has *universal* perfect state transfer if (3) occurs between all distinct pairs of vertices a and b of G.

2.1. *Example: triangle*

We begin by describing an explicit example of the role of weights for perfect state transfer in a triangle, or K_3, which is the complete graph on three vertices. The eigenvalues of K_3 are 2 (simple) and -1 (with multiplicity 2) with eigenvectors $|F_k\rangle$, which are the columns of the Fourier matrix, with $\langle j|F_k\rangle = \omega_3^{jk}/\sqrt{3}$, for $j, k \in$



(a) (b)

Fig. 1. Weighted joins: (a) $K_2 + K_1$ (b) $K_2 + C_4$. Perfect state transfer occurs between the weighted self-loop vertices. Without the self-loops and weights, there is no perfect state transfer.

$\{0, 1, 2\}$.[16] The quantum walk on K_3 yields

$$\langle 1|e^{-itK_3}|0\rangle = \langle 1|\left\{\sum_{k=0}^{2} e^{-it\lambda_k}|F_k\rangle\langle F_k|\right\}|0\rangle = -\frac{2}{3}ie^{-it/2}\sin(3t/2). \tag{4}$$

So, it is clear that there is no perfect state transfer on K_3.[17,1] Now, consider adding self-loops on vertices 0 and 1 with weight μ and putting a weight of η on the edge connecting 0 and 1. The adjacency matrix of this weighted $\tilde{K}_3$ is

$$\tilde{K}_3 = \begin{bmatrix} \mu & \eta & 1 \\ \eta & \mu & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{5}$$

The spectra of $\tilde{K}_3$ is given by the eigenvalues $\lambda_0 = \mu - \eta$ and $\lambda_\pm = 2\alpha_\pm$, where $\alpha_\pm = (1/4)(\delta \pm \Delta)$, $\delta = \mu + \eta$ and $\Delta = \sqrt{\delta^2 + 8}$, with the corresponding orthonormal eigenvectors

$$|v_0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad |v_\pm\rangle = \frac{1}{\sqrt{2\alpha_\pm^2 + 1}}\begin{bmatrix} \alpha_\pm \\ \alpha_\pm \\ 1 \end{bmatrix}. \tag{6}$$

The perfect state transfer equation between the two vertices with weighted self-loops is given by

$$\langle 1|e^{-it\tilde{K}_3}|0\rangle = \langle 1|e^{-it\tilde{K}_2}|0\rangle + \frac{1}{2}e^{-it\delta}\left\{e^{it\delta/2}\left[\cos\left(\frac{\Delta}{2}t\right) - i\frac{\delta}{\Delta}\sin\left(\frac{\Delta}{2}t\right)\right] - 1\right\}, \tag{7}$$

where $\tilde{K}_2$ is $\tilde{K}_2(\mu, \eta)$. Recall that the perfect state transfer $\langle 1|e^{-itK_2}|0\rangle$ on (unweighted) K_2 is given by $-i\sin(t)$. Thus, the weighted $\tilde{K}_2$ has perfect state transfer at time $t^\star = (2\mathbb{Z} + 1)\pi/2\eta$, since the self-loop weight μ disappears into an irrelevant phase factor and the edge weight η translates into a time-scaling. So, to achieve perfect state transfer on $\tilde{K}_3$, it suffices to have

$$\cos\left(\frac{\delta}{4\eta}\pi\right)\cos\left(\frac{\Delta}{4\eta}\pi\right) = 1. \tag{8}$$

Equivalently, we require that:

(1) $A \doteq \delta/4\eta$ be an integer;
(2) $B \doteq \Delta/4\eta$ be an integer; and
(3) $A \equiv B \pmod 2$ or that A and B have the same parity.

From the first two conditions, we require that δ/Δ be a rational number $p/q < 1$ with gcd(p,q)=1. Restating this last condition on p and q and simplifying, we get

$$\delta = p\sqrt{\frac{8}{q^2 - p^2}}, \quad \Delta = q\sqrt{\frac{8}{q^2 - p^2}}. \tag{9}$$

So, we may choose

$$\eta = \frac{1}{4}\sqrt{\frac{8}{q^2 - p^2}}, \tag{10}$$

so that both $\delta/4\eta$ and $\Delta/4\eta$ are integers. Therefore, we choose odd integers p and q satisfying $\gcd(p, q) = 1$; this will satisfy all the three conditions stated above. This shows that there are infinitely many weights μ and η (via infinitely many choices of odd integers p and q) which allow perfect state transfer on $\tilde{K}_3$. We generalize this example in our join theorem for arbitrary regular weighted graphs.

This example complements a result of Casaccino *et al.*[1] which showed the power of weighted self-loops on complete graphs. Our analysis above shows that perfect state transfer is achieved through edge weights instead.

3. Join of Regular Weighted Graphs

In this section, we prove that the existence of perfect state transfer on a join of two arbitrary regular weighted graphs can be reduced to perfect state transfer on one of the graphs. In fact, since we add weights to our graphs in a particular way, this is a reduction onto the unweighted version of one of the graphs. This allows us to analyze the double cone on any regular graph, i.e. the join of K_2 with an arbitrary regular graph. The next theorem is a generalization of a similar join theorem proven by Angeles-Canul *et al.*[12]

Theorem 1. *For $j \in \{1, 2\}$, let $\tilde{G}_j(\mu_j, \eta_j)$ be a k_j-regular graph on n_j vertices, where each vertex has a self-loop with weight μ_j and each edge has weight η_j. Also, for $j \in \{1, 2\}$, let*

$$\kappa_j = \mu_j + \eta_j k_j. \tag{11}$$

Suppose that a and b are two vertices in $\tilde{G}_1$. Let $\mathcal{G} = \tilde{G}_1(\mu_1, \eta_1) + \tilde{G}_2(\mu_2, \eta_2)$ be the join of the weighted graphs. Then,

$$\langle b|e^{-itA_{\mathcal{G}}}|a\rangle = \langle b|e^{-itA_{\tilde{G}_1}}|a\rangle + \frac{e^{-it\kappa_1}}{n_1}\left\{e^{it\delta/2}\left[\cos\left(\frac{\Delta t}{2}\right) - i\left(\frac{\delta}{\Delta}\right)\sin\left(\frac{\Delta t}{2}\right)\right] - 1\right\}, \tag{12}$$

where $\delta = \kappa_1 - \kappa_2$ and $\Delta = \sqrt{\delta^2 + 4n_1 n_2}$.

Proof. Let G_j be the simple and unweighted version of $\tilde{G}_j$, for $j \in \{1,2\}$, i.e. $G_j = \tilde{G}_j(0,1)$. Whenever it is clear from the context, we denote $\tilde{G}_j(\mu_j, \eta_j)$ as simply $\tilde{G}_j$.

If λ_k and $|u_k\rangle$ are the eigenvalues and eigenvectors of A_{G_1}, for $k = 0, \ldots, n_1 - 1$, then

$$\langle b|e^{-itA_{G_1}}|a\rangle = \langle b|\left\{\sum_{k=0}^{n_1-1} |u_k\rangle\langle u_k|e^{-it\lambda_k}\right\}|a\rangle. \tag{13}$$

Here, we assume that $|u_0\rangle$ is the all$-$one eigenvector (which is orthogonal to the other eigenvectors) with eigenvalue $\lambda_0 = k_1$. By the same token, let θ_ℓ and $|v_\ell\rangle$ be the eigenvalues and eigenvectors of A_{G_2}, for $\ell = 0, \ldots, n_2 - 1$. Also, let $|v_0\rangle$ be the all$-$one eigenvector (with eigenvalue $\theta_0 = k_2$) which is orthogonal to the other eigenvectors $|v_\ell\rangle$, $\ell \neq 0$.

Let $\mathcal{G} = \tilde{G}_1 + \tilde{G}_2$. Note that the adjacency matrix of $\mathcal{G}$ is

$$A_{\mathcal{G}} = \begin{bmatrix} \mu_1 I + \eta_1 A_{G_1} & J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & \mu_2 I + \eta_2 A_{G_2} \end{bmatrix}. \tag{14}$$

Let $\delta = \kappa_1 - \kappa_2$, where $\kappa_j = \mu_j + \eta_j k_j$, for $j \in \{1,2\}$. The eigenvalues and eigenvectors of $A_{\mathcal{G}}$ are given by the following three sets:

- For $k = 1, \ldots, n_1 - 1$, let $|u_k, 0_{n_2}\rangle$ be a column vector formed by concatenating the column vector $|u_k\rangle$ with the zero vector of length n_2. Then $|u_k, 0_{n_2}\rangle$ is an eigenvector with eigenvalue $\tilde{\lambda}_k = \mu_1 + \eta_1 \lambda_k$. Note that $\tilde{\lambda}_0 = \kappa_1$.
- For $\ell = 1, \ldots, n_2 - 1$, let $|0_{n_1}, v_\ell\rangle$ be a column vector formed by concatenating the zero vector of length n_1 with the column vector $|v_\ell\rangle$. Then $|0_{n_1}, v_\ell\rangle$ is an eigenvector with eigenvalue $\tilde{\theta}_\ell = \mu_2 + \eta_2 \theta_\ell$.
- Let $|\pm\rangle = 1/\sqrt{L_\pm}|\alpha_\pm, 1_{n_2}\rangle$ be a column vector formed by concatenating the vector $\alpha_\pm|1_{n_1}\rangle$ with the vector $|1_{n_2}\rangle$, where $|1_{n_1}\rangle$, $|1_{n_2}\rangle$ denote the all–one vectors of length n_1, n_2, respectively. Then $|\pm\rangle$ is an eigenvector with eigenvalue $\tilde{\lambda}_\pm = n_1\alpha_\pm + \kappa_2$. Here, we have

$$\alpha_\pm = \frac{1}{2n_1}(\delta \pm \Delta), \quad \Delta^2 = \delta^2 + 4n_1 n_2, \quad L_\pm = n_1(\alpha_\pm)^2 + n_2. \tag{15}$$

In what follows, we will abuse the notation by using $|a\rangle$, $|b\rangle$ for both $\tilde{G}_1$ and $\tilde{G}_1 + \tilde{G}_2$; their dimensions differ in both cases, although it will be clear from the context which version is used. The quantum wave amplitude from a to b is given by

$$\langle b|e^{-itA_{\mathcal{G}}}|a\rangle = \langle b|e^{-itA_{\mathcal{G}}}\left\{\sum_{k=1}^{n_1-1} \langle u_k, 0_{n_2}|a\rangle|u_k, 0_{n_2}\rangle + \sum_{\pm} \frac{\alpha_\pm}{\sqrt{L_\pm}}|\pm\rangle\right\} \tag{16}$$

$$= \langle b|\left\{\sum_{k=1}^{n_1-1} \langle u_k|a\rangle e^{-it\tilde{\lambda}_k}|u_k, 0_{n_2}\rangle + \sum_{\pm} \frac{\alpha_\pm}{\sqrt{L_\pm}} e^{-it\tilde{\lambda}_\pm}|\pm\rangle\right\} \tag{17}$$

$$= \sum_{k=1}^{n_1-1} \langle b|u_k\rangle\langle u_k|a\rangle e^{-it\tilde{\lambda}_k} + \sum_{\pm} \frac{\alpha_\pm^2}{L_\pm} e^{-it\tilde{\lambda}_\pm}. \tag{18}$$

This shows that

$$\langle b|e^{-itA_{\mathcal{G}}}|a\rangle = \langle b|\left\{\sum_{k=0}^{n_1-1} |u_k\rangle\langle u_k|e^{-it\tilde{\lambda}_k}\right\}|a\rangle - \frac{e^{-it\kappa_1}}{n_1} + \sum_{\pm}\frac{\alpha_{\pm}^2}{L_{\pm}}e^{-it\tilde{\lambda}_{\pm}} \tag{19}$$

$$= \langle b|e^{-itA_{\tilde{G}_1}}|a\rangle + \sum_{\pm}\frac{\alpha_{\pm}^2}{L_{\pm}}e^{-it\tilde{\lambda}_{\pm}} - \frac{e^{-it\kappa_1}}{n_1}. \tag{20}$$

To analyze the second term next, we use the following identities whose correctness follows easily from the definitions of $\alpha_{\pm}$, $L_{\pm}$, δ and Δ:

$$\alpha_{+}\alpha_{-} = -(n_2/n_1), \tag{21}$$

$$\alpha_{+} + \alpha_{-} = \delta/n_1, \tag{22}$$

$$L_{+}L_{-} = (n_2/n_1)\Delta^2, \tag{23}$$

$$L_{+} + L_{-} = \Delta^2/n_1, \tag{24}$$

$$(\alpha_{\pm})^2 L_{\mp} = (n_2/n_1)L_{\pm}, \tag{25}$$

$$\tilde{\lambda}_{\pm} = (\hat{\delta} \pm \Delta)/2, \tag{26}$$

where $\hat{\delta} = \kappa_1 + \kappa_2$. Therefore, the summand in (20) is given by

$$\sum_{\pm}\frac{\alpha_{\pm}^2}{L_{\pm}}e^{-it\tilde{\lambda}_{\pm}} = \frac{1}{n_1}e^{-it\hat{\delta}/2}\left[\cos\left(\frac{\Delta t}{2}\right) - i\left(\frac{\delta}{\Delta}\right)\sin\left(\frac{\Delta t}{2}\right)\right]. \tag{27}$$

This yields

$$\langle b|e^{-itA_{\mathcal{G}}}|a\rangle = \langle b|e^{-itA_{\tilde{G}_1}}|a\rangle + \frac{e^{-it\kappa_1}}{n_1}\left\{e^{it\delta/2}\left[\cos\left(\frac{\Delta t}{2}\right) - i\left(\frac{\delta}{\Delta}\right)\sin\left(\frac{\Delta t}{2}\right)\right] - 1\right\}, \tag{28}$$

which proves the claim. □

We describe several applications of Theorem 1 to the weighted double cone $\tilde{K}_2 + G$, for any regular graph G. For notational simplicity, let K_2^b denote K_2 if $b = 1$ and $\overline{K}_2$ if $b = 0$.

Remark. The next corollary complements the observation made by Casaccino *et al.*[1] on $K_2 + K_m$, where each vertex of K_2 has a weighted self-loop. They showed that perfect state transfer occurs in this weighted graph, in contrast to the unweighted version.

Corollary 1. *For any k-regular graph G on n vertices and any $b \in \{0, 1\}$, there exist weights $\mu, \eta \in \mathbb{R}^+$ so that the double cone $\tilde{K}_2^b(\mu, \eta) + G$ has perfect state transfer between the two vertices of $\tilde{K}_2^b$.*

Proof. Consider the weighted double cone $\tilde{K}_2^b(\mu, \eta) + \tilde{G}(0, 1)$, where $\tilde{G}(0, 1)$ is simply the unweighted graph G. We know that $\tilde{K}_2^b(\mu, \eta)$ has perfect state transfer for $b\eta t^{\star} = (2\mathbb{Z} + 1)\pi/2$. Note that when $b = 0$, the perfect state transfer time is ∞ or nonexistent. Let $\delta = (\mu + b\eta) - k$ and $\Delta^2 = \delta^2 + 8n$. By Theorem 1, it suffices to have

$$\cos\left(\frac{\delta}{2}t^\star\right)\cos\left(\frac{\Delta}{2}t^\star\right) = \cos\left(\frac{\delta}{4\eta}\pi\right)\cos\left(\frac{\Delta}{4\eta}\pi\right) = (-1)^{1-b}. \tag{29}$$

So, we require that:

(1) $A \doteq \delta/4\eta$ be an integer;
(2) $B \doteq \Delta/4\eta$ be an integer; and
(3) $[[A \equiv B \pmod 2]] = b$; or that A and B have the same parity if and only if $b = 1$.

From the first two conditions, we require that δ/Δ be a rational number $p/q < 1$ with $\gcd(p, q) = 1$. Restating this last condition on p and q and simplifying, we get

$$\delta = p\sqrt{\frac{8n}{q^2 - p^2}}, \quad \Delta = q\sqrt{\frac{8n}{q^2 - p^2}}. \tag{30}$$

Thus, we may choose

$$\eta = \frac{1}{4}\sqrt{\frac{8n}{q^2 - p^2}}, \tag{31}$$

so that both $\delta/4\eta$ and $\Delta/4\eta$ are integers. Therefore, we choose integers p and q satisfying $\gcd(p, q) = 1$ and $[[p \equiv q \pmod 2]] = b$; this will satisfy all the three conditions stated above. Finally, we may choose $\mu = b\eta - k - \delta$ to complete the weight parameters. □

3.1. *Double half-cones*

In this subsection, we consider graphs obtained by taking a half-join between K_2 and $G + G$, for some arbitrary k-regular graph G, where each vertex of K_2 is connected to only one copy of G in the join $G + G$. When $G = \overline{K}_n$, this half-join is obtained by selecting two adjacent vertices in the complete bipartite graph $K_{n+1,n+1}$. In contrast to complete graphs, we show that weights are not helpful in complete bipartite graphs for achieving perfect state transfer. In fact, we prove a stronger result where perfect state transfer still does not exist even if weights are added to some of the other sets of edges.

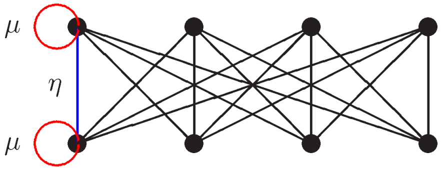


Fig. 2. Weighted half-join between K_2 and $K_{3,3}$. This is equivalent to adding weights to a connected pair of vertices in the complete bipartite graph $K_{4,4}$. There is no perfect state transfer between the two vertices with weighted self-loops.

Theorem 2. *Let G be a k-regular graph on n vertices. Let $\mathcal{G}(\mu, \eta; \kappa, \tau, \rho; \varepsilon)$ be a graph obtained from $\tilde{K}_2(\mu, \eta)$ and $\tilde{G}(\kappa, \tau) +_\rho \tilde{G}(\kappa, \tau)$ by connecting each vertex of $\tilde{K}_2(\mu, \eta)$ to exactly one copy of $\tilde{G}(\kappa, \tau)$ in the weighted join $\tilde{G}(\kappa, \tau) +_\rho \tilde{G}(\kappa, \tau)$ and assigning a weight of ε to each of these connecting edges. Then, there are no nonzero real-valued weights μ, η, κ, τ, ρ or ε for which $\mathcal{G}(\mu, \eta; \kappa, \tau, \rho; \varepsilon)$ has perfect state transfer between the two vertices of $\tilde{K}_2(\mu, \eta)$.*

Remark. Note that if $\varepsilon = 0$, then we have perfect state transfer in $\mathcal{G}$ trivially.

Proof. The adjacency matrix of $\mathcal{G}$ is given by

$$A_{\mathcal{G}} = \begin{bmatrix} \mu & \eta & \varepsilon \mathbf{1}_n^T & \mathbf{0}_n^T \\ \eta & \mu & \mathbf{0}_n^T & \varepsilon \mathbf{1}_n^T \\ \varepsilon \mathbf{1}_n & \mathbf{0}_n & \kappa I_n + \tau A_G & \rho J_n \\ \mathbf{0}_n & \varepsilon \mathbf{1}_n & \rho J_n & \kappa I_n + \tau A_G \end{bmatrix}, \tag{32}$$

where $\mathbf{0}_n$ and $\mathbf{1}_n$ denote the all−zero and all−one column vectors of dimension n, respectively. Suppose that $A_G|u_j\rangle = \lambda_j|u_j\rangle$ are the eigenvalues and eigenvectors of G, for $0 \leq j \leq n-1$, with $|u_0\rangle$ being the all−one eigenvector with $\lambda_0 = k$. Then the spectra of $A_{\mathcal{G}}$ is given by the following sets:

(1) The eigenvectors $|0, 0, 0_n, u_j\rangle$ and $|0, 0, u_j, 0_n\rangle$ share the eigenvalues $\kappa + \tau\lambda_j$, for $1 \leq j \leq n-1$.

(2) Let

$$\alpha_\pm = \frac{1}{2}(\delta_\alpha \pm \Delta_\alpha), \tag{33}$$

where $\delta_\alpha = (\mu + \eta) - (\kappa + \tau k + \rho n)$ and $\Delta_\alpha^2 = \delta_\alpha^2 + 4\varepsilon^2 n$. Then, the two eigenvectors

$$|\alpha_\pm\rangle = \frac{1}{\sqrt{L_\pm^\alpha}}[\,\alpha_\pm \ \alpha_\pm \ 1_n \ 1_n\,]^T \tag{34}$$

have $\lambda_\pm = \alpha_\pm + (\kappa + \tau k + \rho n)$ as eigenvalues. Here $L_\pm^\alpha = 2(\alpha_\pm)^2 + 2n$ is the normalization constant.

(3) Let

$$\beta_\pm = \frac{1}{2}(\delta_\beta \pm \Delta_\beta), \tag{35}$$

where $\delta_\beta = (\mu - \eta) - (\kappa + \tau k - \rho n)$ and $\Delta_\beta^2 = \delta_\beta^2 + 4\varepsilon^2 n$. Then, the two eigenvectors

$$|\beta_\pm\rangle = \frac{1}{\sqrt{L_\pm^\beta}}[\,\beta_\pm \ -\beta_\pm \ 1_n \ -1_n\,] \tag{36}$$

have $\theta_\pm = \beta_\pm + (\kappa + \tau k - \rho n)$ as eigenvalues. Here $L_\pm^\beta = 2(\beta_\pm)^2 + 2n$ is the normalization constant.

The following identities can be verified easily: for $\xi \in \{\alpha, \beta\}$, we have

$$L_+^{\xi} L_-^{\xi} = 4n\Delta_{\xi}^2/\varepsilon^2, \tag{37}$$

$$\xi_+ \xi_- = -n, \tag{38}$$

$$\xi_{\pm}^2 L_{\mp}^{\xi} = nL_{\pm}^{\xi}. \tag{39}$$

Using these, the quantum walk on $\mathcal{G}$ starting at a and ending at b is given by

$$\langle b|e^{-itA_{\mathcal{G}}}|a\rangle = \left\{\sum_{\pm} e^{-it\lambda_{\pm}} \frac{\alpha_{\pm}^2}{L_{\pm}^{\alpha}}\right\} - \left\{\sum_{\pm} e^{-it\theta_{\pm}} \frac{\beta_{\pm}^2}{L_{\pm}^{\beta}}\right\}. \tag{40}$$

After simplifications, we obtain

$$\langle b|e^{-itA_{\mathcal{G}}}|a\rangle = \frac{e^{-i(\kappa+\tau k)t}}{2} e^{-i(\rho n)t} e^{-i\delta_{\alpha} t/2}\left[\cos\left(\frac{\Delta_{\alpha}}{2}t\right) - i\frac{\delta_{\alpha}}{\Delta_{\alpha}}\sin\left(\frac{\Delta_{\alpha}}{2}t\right)\right] \tag{41}$$

$$-\frac{e^{-i(\kappa+\tau k)t}}{2} e^{i(\rho n)t} e^{-i\delta_{\beta} t/2}\left[\cos\left(\frac{\Delta_{\beta}}{2}t\right) - i\frac{\delta_{\beta}}{\Delta_{\beta}}\sin\left(\frac{\Delta_{\beta}}{2}t\right)\right]. \tag{42}$$

Ignoring the irrelevant phase factor $e^{-i(\kappa+\tau k)t}$ and noting that the damping factor δ/Δ forces the sine term to vanish, we get

$$\begin{aligned}\langle b|e^{-itA_{\mathcal{G}}}|a\rangle &= \frac{e^{-i(\rho n)t}}{2}\cos\left(\frac{\delta_{\alpha}}{2}t\right)\cos\left(\frac{\Delta_{\alpha}}{2}t\right)\\ &\quad -\frac{e^{i(\rho n)t}}{2}\cos\left(\frac{\delta_{\beta}}{2}t\right)\cos\left(\frac{\Delta_{\beta}}{2}t\right).\end{aligned} \tag{43}$$

We choose $t^{\star}$ so that $e^{-i(\rho n)t^{\star}} = 1$, which implies that $t^{\star} = 2\mathbb{Z}\pi/\rho n$. This simplifies the above expression to

$$\langle b|e^{-it^{\star}A_{\mathcal{G}}}|a\rangle = \frac{1}{2}\cos\left(\frac{\delta_{\alpha}}{2}t^{\star}\right)\cos\left(\frac{\Delta_{\alpha}}{2}t^{\star}\right) - \frac{1}{2}\cos\left(\frac{\delta_{\beta}}{2}t^{\star}\right)\cos\left(\frac{\Delta_{\beta}}{2}t^{\star}\right). \tag{44}$$

For simplicity, define

$$Z_{\alpha} = \cos\left(\frac{\delta_{\alpha}}{2}t^{\star}\right)\cos\left(\frac{\Delta_{\alpha}}{2}t^{\star}\right) = \cos\left(\frac{\delta_{\alpha}}{\rho n}\pi\right)\cos\left(\frac{\Delta_{\alpha}}{\rho n}\pi\right), \tag{45}$$

$$Z_{\beta} = \cos\left(\frac{\delta_{\beta}}{2}t^{\star}\right)\cos\left(\frac{\Delta_{\beta}}{2}t^{\star}\right) = \cos\left(\frac{\delta_{\beta}}{\rho n}\pi\right)\cos\left(\frac{\Delta_{\beta}}{\rho n}\pi\right). \tag{46}$$

Let

$$\tilde{P}_{\alpha} = \frac{\delta_{\alpha}}{\rho n}, \quad \tilde{Q}_{\alpha} = \frac{\Delta_{\alpha}}{\rho n}, \quad \tilde{P}_{\beta} = \frac{\delta_{\beta}}{\rho n}, \quad \tilde{Q}_{\beta} = \frac{\Delta_{\beta}}{\rho n}. \tag{47}$$

To achieve perfect state transfer, we require that $Z_{\alpha}Z_{\beta} = -1$. For example, if we require that $Z_{\alpha} = -1$ and $Z_{\beta} = 1$, then it suffices to impose the following *integrality* and *parity* conditions:

$$\tilde{P}_{\alpha}, \tilde{Q}_{\alpha} \in \mathbb{Z}, \quad \tilde{P}_{\alpha} \not\equiv \tilde{Q}_{\alpha} \pmod 2, \tag{48}$$

$$\tilde{P}_\beta, \tilde{Q}_\beta \in \mathbb{Z}, \quad \tilde{P}_\beta \equiv \tilde{Q}_\beta \pmod{2}. \tag{49}$$

We will show that there is no ρ which can satisfy all the above conditions.

Suppose that, for $\xi \in \{\alpha, \beta\}$, we have

$$\frac{\delta_\xi}{\Delta_\xi} = \frac{p_\xi}{q_\xi} \in \mathbb{Q}, \tag{50}$$

where p_ξ and q_ξ are integers with $\gcd(p_\xi, q_\xi) = 1$; moreover, since $\Delta_\xi^2 = \delta_\xi^2 + 4\varepsilon^2 n$, we get

$$\delta_\xi = p_\xi \sqrt{\frac{4\varepsilon^2 n}{q_\xi^2 - p_\xi^2}}, \quad \Delta_\xi = q_\xi \sqrt{\frac{4\varepsilon^2 n}{q_\xi^2 - p_\xi^2}}. \tag{51}$$

Consider $\tilde{P}_\xi$ and $\tilde{Q}_\xi$, for $\xi \in \{\alpha, \beta\}$. Letting $\Lambda = 2\varepsilon/\rho\sqrt{n}$, we have

$$\tilde{P}_\xi = p_\xi \frac{\Lambda}{\sqrt{q_\xi^2 - p_\xi^2}}, \quad \tilde{Q}_\xi = q_\xi \frac{\Lambda}{\sqrt{q_\xi^2 - p_\xi^2}}. \tag{52}$$

Since $\tilde{P}_\alpha \equiv \tilde{P}_\alpha^2 \pmod{2}$, we know that $\tilde{P}_\alpha^2 \not\equiv \tilde{Q}_\alpha^2 \pmod{2}$ is equivalent to $\tilde{P}_\alpha \not\equiv \tilde{Q}_\alpha \pmod{2}$. Likewise, $\tilde{P}_\beta^2 \equiv \tilde{Q}_\beta^2 \pmod{2}$ is equivalent to $\tilde{P}_\beta \equiv \tilde{Q}_\beta \pmod{2}$. This changes (52) to

$$\tilde{P}_\xi^2 = p_\xi^2 \frac{\Lambda^2}{q_\xi^2 - p_\xi^2}, \quad \tilde{Q}_\xi^2 = q_\xi^2 \frac{\Lambda^2}{q_\xi^2 - p_\xi^2}. \tag{53}$$

Since we require that $\tilde{P}_\xi$ and $\tilde{Q}_\xi$ be integers, $(q_\xi^2 - p_\xi^2) \mid q_\xi^2 \Lambda^2$ and $(q_\xi^2 - p_\xi^2) \mid p_\xi^2 \Lambda^2$. However, $gcd(p_\xi, q_\xi) = 1$ implies that $gcd(p_\xi^2, q_\xi^2) = 1$. This gives us $(q_\xi^2 - p_\xi^2) \mid \Lambda^2$.

Suppose now that $p_\beta^2 \equiv q_\beta^2 \pmod{2}$. Then $q_\beta^2 - p_\beta^2$ is even. This forces Λ^2 to be even. Similarly, suppose that $p_\beta^2 \not\equiv q_\beta^2 \pmod{2}$. Then $q_\beta^2 - p_\beta^2$ is odd. However, since $\tilde{P}_\beta^2 \equiv \tilde{Q}_\beta^2 \pmod{2}$ and one of p_β^2, q_β^2 is odd, Λ^2 must be even.

In both cases, Λ^2 is even. Allowing $p_\alpha^2 \equiv q_\alpha^2 \pmod{2}$ guarantees that $\tilde{P}_\alpha^2 \equiv \tilde{Q}_\alpha^2 \pmod{2}$. Letting $p_\alpha^2 \not\equiv q_\alpha^2 \pmod{2}$ gives us $q_\alpha^2 - p_\alpha^2$ to be odd. This again forces $\tilde{P}_\alpha^2 \equiv \tilde{Q}_\alpha^2 \pmod{2}$. Both instances contradict our given requirement that $\tilde{P}_\alpha^2 \not\equiv \tilde{Q}_\alpha^2 \pmod{2}$.

The case where we require that $Z_\alpha = 1$ and $Z_\beta = -1$, i.e. where $\tilde{P}_\alpha$ is even and $\tilde{Q}_\alpha$, $\tilde{P}_\beta$, $\tilde{Q}_\beta$ are odd, may be treated similarly. □

Corollary 2. *For any $n \geq 2$, consider the complete bipartite graph $K_{n,n}$. Let a and b be two arbitrary adjacent vertices in $K_{n,n}$. Then, there are no self-loop weights μ on a and b and edge weight η on the edge (a, b) for which there is perfect state transfer from vertex a to vertex b in this weighted version of $K_{n,n}$.*

Proof. We apply Theorem 2 with $G = \overline{K}_{n-1}$ set to the empty graph on $n - 1$ vertices, i.e. is A_G is the all$-$zero matrix and hence $k = 0$. Also, we set $\varepsilon = 1$, $\kappa = 0$ and τ is an arbitrary value. In the proof of Theorem 2, setting $\kappa = 0$ does not affect perfect state transfer since the term $\kappa + k\tau$ may be ignored due to its contribution as a global phase factor. Setting $\varepsilon = 1$ does not affect perfect state transfer since it is "factored out" through Λ. Thus, these specific settings of values do not affect the conclusions of Theorem 2. $\square$

4. Hamming Graphs

We show that the class of weighted Hamming graphs exhibit perfect state transfer between any two of their vertices. First, we prove the following closure result on the Cartesian product of graphs. This is an adaptation of a similar theorem for the unweighted case.[12]

Theorem 3. *For $j = 1, \ldots, m$, the graph G_j has perfect state transfer from a_j to b_j at time t_j if and only if $\mathcal{G} = \oplus_{j=1}^{m} \tilde{G}_j(\mu_j, \eta_j)$ has perfect state transfer from $(a_1, \ldots, a_m)$ to $(b_1, \ldots, b_m)$ at time $t^\star$, whenever $t^\star = t_j/\eta_j$. This holds independently of the choice of the self-loop weights μ_j.*

Proof. We prove the claim for $m = 2$. Suppose that the unweighted graph G_j has perfect state transfer from a_j to b_j at time $t_j^\star$. Consider the quantum walk on the $\tilde{G}_1(\mu_1, \eta_1) \oplus \tilde{G}_2(\mu_2, \eta_2)$. In shorthand, we denote each graph simply as $\tilde{G}_j$:

$$\begin{aligned}
\langle b_1, b_2 | e^{-itA_{\tilde{G}_1 \oplus \tilde{G}_2}} | a_1, a_2 \rangle &= \langle b_1 | \langle b_2 | e^{-it(I \otimes A_{\tilde{G}_2})} e^{-it(A_{\tilde{G}_1} \otimes I)} | a_1 \rangle | a_2 \rangle \quad (54)\\
&= \langle b_1 | \langle b_2 | (I \otimes e^{-itA_{\tilde{G}_2}})(e^{-itA_{\tilde{G}_1}} \otimes I) | a_1 \rangle | a_2 \rangle \quad (55)\\
&= \langle b_1 | e^{-itA_{\tilde{G}_1}} | a_1 \rangle \langle b_2 | e^{-itA_{\tilde{G}_2}} | a_2 \rangle. \quad (56)
\end{aligned}$$

Since $A_{\tilde{G}(\mu,\eta)} = \mu I + \eta A_G$, we have

$$\langle b | e^{-itA_{\tilde{G}}} | a \rangle = e^{-i\mu t} \langle b | e^{-i\eta t A_G} | a \rangle. \quad (57)$$

Therefore, the quantum walk on the weighted Cartesian product yields

$$\langle b_1, b_2 | e^{-itA_{\tilde{G}_1 \oplus \tilde{G}_2}} | a_1, a_2 \rangle = e^{-i(\mu_1+\mu_2)t} \langle b_1 | e^{-i\eta_1 t A_{G_1}} | a_1 \rangle \langle b_2 | e^{-i\eta_2 t A_{G_2}} | a_2 \rangle. \quad (58)$$

This shows that $\tilde{G}_1 \oplus \tilde{G}_2$ has perfect state transfer from (a_1, a_2) to (b_1, b_2) at time t if and only if G_1 has perfect state transfer from a_1 to b_1 at time $\eta_1 t$ and G_2 has perfect

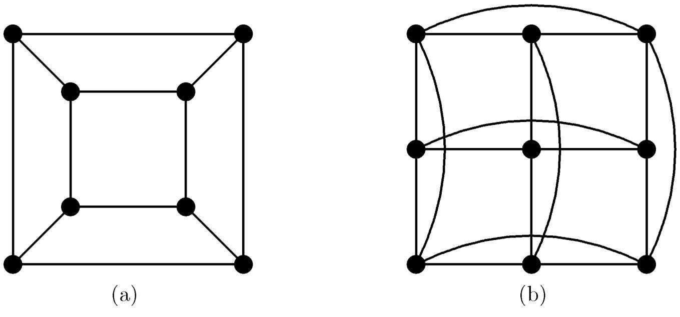

(a) (b)

Fig. 3. Hamming graphs: (a) $H(2,3)$, (b) $H(3,2)$. Perfect state transfer occurs between any pair of vertices with the help of weighted self-loops and edges.

state transfer from a_2 to b_2 at time $\eta_2 t$. So, if the weights η_j satisfy $\eta_j t^\star = t_j$, for all j, then $\tilde{G}_1 \oplus \tilde{G}_2$ has perfect state transfer at time $t^\star$. The general claim follows by induction. □

Theorem 4. *The class $\tilde{H}(q, n)$ of weighted Hamming graphs has universal perfect state transfer at an arbitrarily chosen time.*

Proof. Recall that $H(q, n) = K_q^{\oplus n}$. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two vertices of $\tilde{H}(q, n)$. By Corollary 1, we know that $\tilde{K}_q$ has perfect state transfer between any two of its vertices for a suitable choice of weights. For each dimension $j \in \{1, \ldots, n\}$, fix a set of weights so that $\tilde{K}_q^{(j)}$ has perfect state transfer from a_j to b_j. Then, by Theorem 3, $\oplus_{j=1}^{n} \tilde{K}_q^{(j)}$ has perfect state transfer from a to b. □

4.1. *Hypercubes*

In this subsection, we show that a weighted hypercube has the property of universal perfect state transfer. In fact, we prove a stronger statement as given in the next theorem. But, first, we need to define a particular notion of uniform superposition over the n cube.

Fact 5.[3,2] The following facts are known about a quantum walk on the hypercube Q_n at times $t \in \{\pi/4, \pi/2\}$:

$$\langle b|e^{-itQ_n}|a\rangle = \begin{cases} (-i)^{|a\oplus b|}/\sqrt{2^n} & \text{if } t = \pi/4, \\ [[a \oplus b = 1_n]] & \text{if } t = \pi/2. \end{cases} \tag{59}$$

We say that a superposition $|\varrho_n\rangle$ over Q_n is in *normal form* if

$$|\varrho_n\rangle = \frac{1}{\sqrt{2^n}} \sum_{a\in\{0,1\}^n} (-i)^{|a|}|a\rangle. \tag{60}$$

Note that $|\varrho_n\rangle$ is the uniform superposition of a quantum walk on Q_n from 0_n at time $\pi/4$, i.e. $|\varrho_n\rangle = \exp(-i(\pi/4)Q_n)|0_n\rangle$.

Theorem 6. *For any $n \geq 1$, given any two distinct subcubes B_1 and B_2 of Q_n, there is a set of edge weights w so that Q_n^w has perfect state transfer between uniform superpositions in normal form on B_1 and B_2.*

Proof. First, we show that the hypercube Q_n has perfect state transfer from any vertex to any subcube. Since Q_n is vertex-transitive, it suffices to show perfect state transfer from the vertex 0_n to the subcube $B = (1_k 0_\ell \star_m)$, where $m = n - k - \ell$. Define the adjacency matrix of $\tilde{Q}_n$ as

$$\tilde{Q}_n = Q_k \otimes I_{2^{n-k}} + \frac{1}{2} I_{2^{k+\ell}} \otimes Q_m, \tag{61}$$

which is a sum of two commuting matrices. Then, letting $t^\star = \pi/2$, we have

$$\begin{aligned} &\langle 1_k 0_\ell | \langle \varrho_m | \exp\left(-it^\star \tilde{Q}_n\right) | 0_k 0_\ell 0_m \rangle \\ &\quad = \langle 1_k 0_\ell | \langle \varrho_m | \exp\left(-i\frac{t^\star}{2} I_{2^{k+\ell}} \otimes Q_m\right) | 1_k 0_\ell 0_m \rangle. \end{aligned} \tag{62}$$

The equality and the fact that the last expression has unit magnitude follow from Fact 5.

To show perfect state transfer between two arbitrary subcubes, note that we just showed that $|B\rangle = e^{-it^\star \tilde{Q}_n}|0_n\rangle$. Thus, we also have $|0_n\rangle = e^{-it^\star(-\tilde{Q}_n)}|B\rangle$. This proves the claim. □

We recover the result of Bernasconi *et al.*,[2] which we restate in the next corollary, via the use of explicit edge weights on the hypercube.

Corollary 3. *For any $n \geq 1$, given any two distinct vertices a and b of the hypercube Q_n, there is a set of edge weights w so that Q_n^w has perfect state transfer from a to b at time $t^\star = \pi/2$.*

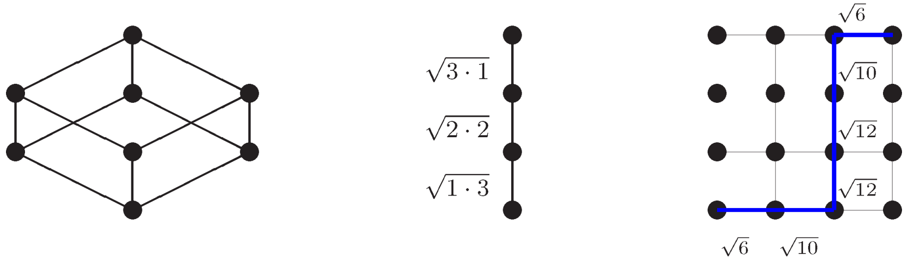
Remark. We note that Bernasconi *et al.*[2] proved universal perfect state transfer for the n cube by *dynamically* changing the underlying hypercubic structure of the graph. In contrast, our scheme is based on using *static* weights which can be interpreted dynamically with time. In both schemes, it is possible to route information through a Hamiltonian path which visits each vertex once and exactly once. We believe that this Hamiltonian property might be of interest in further applications of perfect state transfer.

5. Conclusion

We studied perfect state transfer on quantum networks represented by weighted graphs. Our goal was to understand the role of weights in achieving perfect state transfer on graphs.

First, we proved a join theorem for weighted regular graphs and found, as a corollary, that a weighted double cone on any regular graph has perfect state transfer. This implies as a corollary a result of Casaccino *et al.*[1] where the regular graph is a complete graph. In contrast, we also showed that weights do not help in achieving perfect state transfer on complete bipartite graphs. This is obtained as part of a more general result on graphs constructed from a half-join of K_2 and $G + G$, for an arbitrary regular graph G. We found it curious that the full join connection seemed crucial for weights to have a positive effect in achieving perfect state transfer. We leave the case of complete multipartite graphs and strongly regular graphs as an open question.

Second, we observed that Hamming graphs have the property of universal perfect state transfer. This is a stronger requirement than the standard perfect state transfer property where perfect state transfer must occur between any pair of vertices. Prior



(a) (b) (c)

Fig. 4. Existence of universal near-perfect state transfer on any weighted graph. (a) Q_n has vertex-to-vertex PST.[2] (b) Hypercubic-weighted P_n has end-to-end PST.[4] (c) Emulate the hypercubic weighting along any path between the source and target vertices while setting other weights to near zero.

to this work, the only known family of graphs with universal perfect state transfer were the (unweighted) hypercubic graphs.[2] We proved our result on the Hamming graphs by showing a closure result for a weighted Cartesian product of perfect state tranfer graphs, even when the graph components have different perfect state transfer times. The unweighted version of this closure result, as shown by Angeles-Canul *et al.*,[12] requires a global common perfect state transfer time for all graphs in the Cartesian product. For the hypercubes, we showed a stronger property of universal perfect state transfer, where perfect state transfer occurs between uniform superpositions of two arbitrary subcubes. We imposed a mild condition on the uniform superpositions which exhibit perfect state transfer.

Note that if zero weights are allowed, then universal perfect state transfer is trivial. Simply take any path connecting the two vertices and assign the hypercubic weights to the edges on the path (as in Ref. 4) and zero weights to all other edges. If zero weights are not allowed, then we conjecture that near-perfect state transfer is possible by assigning weights that tend to zero (for the edges which require zero weights).

Acknowledgments

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References

1. A. Casaccino *et al.*, Quantum state transfer through a qubit network with energy shifts and fluctuations [arXiv:quant-ph/0904.4510].
2. A. Bernasconi, C. Godsil and S. Severini, *Phys. Rev. A* **78** (2008) 052320.
3. C. Moore and A. Russell, Quantum walks on the hypercube, in *Proc. 6th Int. Workshop on Randomization and Approximation in Computer Science*, Lecture Notes in Computer Science, Vol. 2483 (Springer, 2002), pp. 164–178.
4. M. Christandl *et al.*, *Phys. Rev. Lett.* **92** (2004) 187902.
5. M. Christandl *et al.*, *Phys. Rev. A* **71** (2005) 032312.

6. N. Saxena, S. Severini and I. Shparlinski, *Int. J. Quant. Inf.* **5**(3) (2007) 417−430.
7. S. Bose *et al.*, *Int. J. Quant. Inf.* **7**(4) (2009) 713−723.
8. M. Bašić and M. Petković, *Appl. Math. Lett.* (in press).
9. E. Farhi and S. Gutmann, *Phys. Rev. A* **58** (1998) 915−928.
10. V. Kendon, *Int. J. Quant. Inf.* **4**(5) (2006) 791−805.
11. D. L. Feder, *Phys. Rev. Lett.* **97** (2006) 180502.
12. R. J. Angeles-Canul *et al.*, Perfect state transfer, integral circulants and join of graphs [arXiv:quant-ph/0907.2148].
13. R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, 2nd edn. (Addison-Wesley, USA, 1994).
14. L. Lovász, *Combinatorial Problems and Exercises*, 2nd edn. (American Mathematical Society, 2007).
15. A. J. Schwenk and R. J. Wilson, Eigenvalues of Graphs, in *Selected Topics in Graph Theory*, eds. Lowell W. Beineke and Robin J. Wilson (Academic Press, USA, 1978), pp. 307−336.
16. N. Biggs, *Algebraic Graph Theory*, 2nd ed. (Cambridge University Press, USA, 1993).
17. A. Ahmadi *et al.*, *Quant. Inf. Comput.* **3**(6) (2003) 611−618.