Vectors and Tensors

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Good Sources

R. Aris, **Vectors, Tensors, and the Equations of Fluid Mechanics**, Prentice Hall (1962).

and Appendices in

(i) R. B. Bird, W. E. Stewart, and E. N. Lightfoot, **Transport Phenomena**, John Wiley, 2002.

(ii) J. Slattery, **Momentum, Energy, and Mass Transfer in Continua**, McGraw-Hill, 1972.

Some Basics

We encounter physical entities such as position, velocity, momentum, stress, temperature heat flux, concentration, and mass flux in transport problems - there is a need to describe them in mathematical terms and manipulate the representations in various ways. This requires the tools of tensor analysis.

Scalars

An entity such as temperature or concentration that has a magnitude (and some units that need not concern us right now), but no sense of direction, is represented by a scalar.

Vectors

In contrast, consider the velocity of a particle or element of fluid; to describe it fully, we need to specify both its magnitude (in some suitable units) and its instantaneous spatial direction. Other examples are momentum, heat flux, and mass flux. These quantities are described by vectors. In books, vectors are printed in boldface. In ordinary writing, we may represent a vector in different ways.

$$\underline{v}, \bar{v}, \vec{v} \qquad \text{or} \qquad v_i$$

Gibbs notation **index notation**

The last requires comment. Whereas we represent the vectorial quantity with a symbol, we often know it only via its components in some basis set. Note that the vector as an entity has an **invariant** identity independent of the basis set in which we choose to represent it.

In index notation, the subscript "i" is a free index - that is, it is allowed to take on any of the three values 1, 2, 3, in 3-dimensional space. Thus, v_i really stands for the ordered set $\left(v_1, v_2, v_3\right)$.

Basis Sets

The most common basis set in three-dimensional space is the orthogonal triad $\left(\underline{i}, \underline{j}, \underline{k}\right)$ corresponding to a rectangular Cartesian coordinate system. $\underline{i}$ stands for a unit vector in the $x-$direction and $\underline{j}$ and $\underline{k}$ represent unit vectors in the y and $z-$directions respectively. Note that this is not a unique basis set. The directions of $\underline{i}, \underline{j}, \underline{k}$ depend on our choice of the coordinate directions.

There is no reason for the basis set to be composed of orthogonal vectors. The only requirement is that the three vectors chosen do not lie in a plane. Orthogonal sets are the most convenient, however.

We find the components of a vector in the directions of the base vectors by taking inner (dot) products.

$$v_x = \underline{v} \cdot \underline{i}, \quad v_y = \underline{v} \cdot \underline{j}, \quad v_z = \underline{v} \cdot \underline{k}$$

Then, $\underline{v} = v_x \underline{i} + v_y \underline{j} + v_z \underline{k}$

You can verify the consistency of the above by taking inner products of both sides of the equation with the base vectors and recognizing that the base vectors are orthogonal.

$$\underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{k} = \underline{k} \cdot \underline{i} \equiv 0$$

The order of the vectors in the inner product is unimportant.

$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$$

Scalar and Vector Fields

In practice the temperature, velocity, and concentration in a fluid vary from point to point (and often with time). Thus, we think of fields - temperature field, velocity field, etc.

In the case of a vector field such as the velocity in a fluid, we need to represent the velocity at every point in space in the domain of interest. The advantage of the rectangular Cartesian basis set $\left(\underline{i}, \underline{j}, \underline{k}\right)$ is that it is invariant as we translate the triad to any point in space. That is, not only are these base vectors of unit length, but they never change direction as we move from one point to another, once we have chosen our $x, y,$ and z directions.

Vector Operations

The entity $\underline{v}$ has an identity of its own. Its length and spatial direction are independent of the basis set we choose. As the vectors in the basis set change, the **components** of $\underline{v}$ change according to standard rules.

Vectors can be added; the results are new vectors. If we use component representation, we simply add each component. Subtraction works in a similar manner.

Vectors also can be multiplied, but there are two ways to do it. We define the dot and cross products, also known as **inner** or **scalar** and **vector** products, respectively, as shown below.

$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$ is a scalar. We commonly use a numerical subscript for the components; in this case, the basis set is the orthogonal triad $\left(\underline{e}_{(1)}, \underline{e}_{(2)}, \underline{e}_{(3)}\right)$. Let

$$\underline{a} = a_1 \underline{e}_{(1)} + a_2 \underline{e}_{(2)} + a_3 \underline{e}_{(3)}$$

Then,

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= \sum_{i=1}^{3} a_i b_i$$

In the above, we usually omit $\sum_{i=1}^{3}$. When an index is repeated, summation over that index is implied.

$\underline{a} \cdot \underline{b} = a_i b_i$ This is called the summation convention

$$\underline{a} \cdot \underline{a} = a_i a_i = a^2 \text{ or } \left|\underline{a}\right|^2$$

where a is the length of $\underline{a}$ and is **invariant**; "invariant" means that the entity does not change as the basis set is altered.

$\underline{a} \times \underline{b}$ is the vector product. As implied by the name, it is a vector; it is normal to the plane containing $\underline{a}$ and $\underline{b}$. ($\underline{a}$, $\underline{b}$, $\underline{a} \times \underline{b}$) form a right-handed system (this is an arbitrary convention , but we have to choose one or the other, so we choose "right"). The order is important, for,

$$\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$$

that is, $\underline{b} \times \underline{a}$ points opposite to $\underline{a} \times \underline{b}$.

We can write

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{e}_{(1)} & \underline{e}_{(2)} & \underline{e}_{(3)} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

There is a compact representation of a determinant that helps us write

$$\underline{a} \times \underline{b} = \varepsilon_{ijk}\, a_i b_j$$

(Note that k is a free index. The actual symbol chosen for it is not important; what matters is that the right side has **one** free index, making it a vector)

ε_{ijk} is called the permutation symbol

$\varepsilon_{ijk} = 0$ if any two of the indices are the same

$= +1$ if i, j, k form an even permutation of 1, 2, 3 [example: 1,2,3]

$= -1$ if i, j, k form an odd permutation of 1, 2, 3 [example: 2, 1, 3]

We can assign a geometric interpretation to $\underline{a} \cdot \underline{b}$ and $\underline{a} \times \underline{b}$. If the angle between the two vectors $\underline{a}$ and $\underline{b}$ is θ, then

$$\underline{a} \cdot \underline{b} = a\ b \cos\theta$$

and the length of $\underline{a} \times \underline{b}$ is $a b \sin\theta$. You may recognize $a\ b \sin\theta$ as the area of the parallelogram formed by $\underline{a}$ and $\underline{b}$ as two adjacent sides. Given this, it is straightforward to see that

$$\underline{a} \cdot \underline{b} \times \underline{c} = \varepsilon_{ijk}\, a_i b_j c_k$$

is the volume of the parallelepiped with sides $\underline{a}, \underline{b}$, and $\underline{c}$. This is called the triple scalar product.

Second Order Tensors

Note that we did not define vector division. The closest we come is in the definition of second-order tensors!

Imagine

$$\frac{\underline{a}}{\underline{b}} = \underline{\underline{\mathrm{T}}}$$

Instead, we write

$$\underline{a} = \underline{\underline{\mathrm{T}}} \cdot \underline{b}$$

A tensor (unless explicitly stated otherwise we'll only be talking about "second-order" and shall therefore omit saying it every time) "operates" on a vector to yield another vector. It is very useful to think of tensors as operators as you'll see later.

Note the "dot" product above. Using ideas from vectors, we can see how the above equation may be written in index notation.

$$a_i = \mathrm{T}_{ij} b_j$$

It is important to note that $\underline{b} \cdot \underline{\underline{\mathrm{T}}}$ would be $b_i \mathrm{T}_{ij}$ and would be different from $\underline{\underline{\mathrm{T}}} \cdot \underline{b}$ in general.

The two underbars in $\underline{\underline{\mathrm{T}}}$ now take on a clear significance; we are referring to a doubly subscripted entity. We can think of a tensor as a sum of components in the same way as a vector. For this, we use the following result.

$$\underline{e}_{(i)} \cdot \underline{\underline{\mathrm{T}}} \cdot \underline{e}_{(j)} = \mathrm{T}_{ij}$$

Scalar

We're not using index notation here

Thus, to get T_{23} we would find $\underline{e}_{(2)} \cdot \underline{\underline{\mathrm{T}}} \cdot \underline{e}_{(3)}$. We can then think of T as a sum.

$$\underline{\underline{\mathrm{T}}} = \mathrm{T}_{11}\, \underline{e}_{(1)}\underline{e}_{(1)} + \mathrm{T}_{12}\, \underline{e}_{(1)}\underline{e}_{(2)} + \mathrm{T}_{13}\, \underline{e}_{(1)}\underline{e}_{(3)}$$

$$+\mathrm{T}_{21}\, \underline{e}_{(2)}\underline{e}_{(1)} + \mathrm{T}_{22}\, \underline{e}_{(2)}\underline{e}_{(2)} + \mathrm{T}_{23}\, \underline{e}_{(2)}\underline{e}_{(3)}$$

$$+\mathrm{T}_{31}\, \underline{e}_{(3)}\underline{e}_{(1)} + \mathrm{T}_{32}\, \underline{e}_{(3)}\underline{e}_{(2)} + \mathrm{T}_{33}\, \underline{e}_{(3)}\underline{e}_{(3)}$$

What are the quantities $\underline{e}_{(1)}\underline{e}_{(2)}$ and others like them? They are called **dyads**. They are a basis set for representing tensors. Each is a tensor that only has one component in this basis set. Note that $\underline{e}_{(i)}\underline{e}_{(j)} \neq \underline{e}_{(j)}\underline{e}_{(i)}$.

You can see that tensors and matrices have a lot in common!

In fact, we commonly write the components of a tensor as the elements of a 3 x 3 matrix.

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

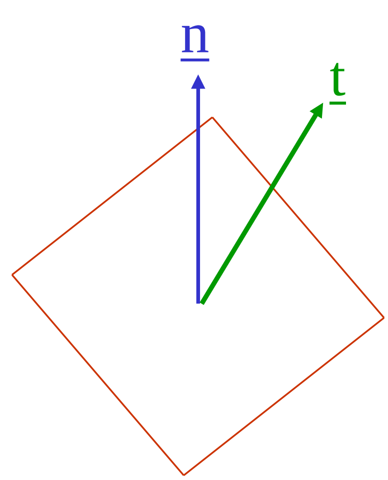
Naturally, as we change our basis set, the components of a given tensor will change, but the entity itself does not change. Of course, unlike vectors, we cannot visualize tensors – we only "know" them by what they do to vectors that we "feed" them!

A good example of a tensor in fluid mechanics is the stress at a point. To completely specify the stress vector, we not only need to specify the point, but also the orientation of the area element. At a given point, we can orient the area in infinitely many directions, and for each orientation, the stress vector would, in general be different.

$$\text{Stress} = \frac{\text{Force}}{\text{Area}} \quad \begin{matrix} \leftarrow \text{has magnitude and direction} \\ \leftarrow \text{has magnitude and direction} \end{matrix}$$

In fact, we can show that stress is indeed a tensor (for proof, see Aris, p. 101). So, we get

$$\underline{t} = \underline{n} \cdot \underline{\underline{T}}$$



The symbol $\underline{n}$ represents the unit normal (vector) to the area element, and $\underline{t}$ is the stress vector acting on that element. The second-order tensor $\underline{\underline{T}}$ completely describes the state of stress at a point. By convention, $\underline{t}$ is the stress exerted **by** the fluid into which $\underline{n}$ points **on** the fluid adjoining it.

Just as a vector has one invariant (its length), a tensor has three invariants. They are defined as follows.

Let $\underline{\underline{A}}$ or A_{ij} be the tensor.

$$I_A = \text{trace}\left\{\underline{\underline{A}}\right\} = \overset{\downarrow \text{ abbreviation}}{tr}\left\{\underline{\underline{A}}\right\} = A_{ii}$$

$$II_A = \frac{1}{2}\left[\left|I_A\right|^2 - \overline{II}_A\right]$$

where

$$\overline{II}_A = tr\left\{\underline{\underline{A}} \cdot \underline{\underline{A}}\right\} \qquad \text{Note: } \underline{\underline{A}} \cdot \underline{\underline{A}} \text{ is the tensor } A_{ij}A_{jk}$$

$$III_A = \text{ Determinant of } \underline{\underline{A}} = \text{Det}\left\{\underline{\underline{A}}\right\}$$

$$= \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$$

$$= \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}$$

As the basis set is changed, the invariants do not change even though the components of the tensor may change. For more details, consult Aris, p. 26, 27 or Slattery, p. 47, 48.

A **symmetric** tensor A_{ij} is one for which $A_{ij} = A_{ji}$. Thus, there are only six independent components. Stress is a symmetric tensor (except in unusual fluids). Symmetric tensors with real elements are self-adjoint operators, a concept about which you can learn more in advanced work.

A **skew-symmetric** tensor A_{ij} is one for which $A_{ij} = -A_{ji}$. You can see immediately that the diagonal elements must be zero $\left(\text{because } A_{ii} = -A_{ii}\right)$. Skew-symmetric tensors have only three independent components. **Vorticity** is an example of a skew symmetric tensor.

If we write a skew-symmetric tensor A_{ij} in the form

$$\begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}$$

We can see that there is a vector $\underline{a}$ that can be formed using the elements of A_{ij}. The two are related by the following result, which is useful in the context of the physical significance of vorticity.

$$\underline{\underline{A}} \cdot \underline{x} = \underline{x} \times \underline{a}$$

Any second order tensor can be decomposed into the sum of a symmetric tensor and a skew-symmetric tensor.

$$A_{ij} = \underbrace{\frac{1}{2}\left(A_{ij} + A_{ji}\right)}_{\text{Symmetric Tensor}} + \underbrace{\frac{1}{2}\left(A_{ij} - A_{ji}\right)}_{\text{Skew-Symmetric Tensor}} \quad \text{or in Gibbs notation,} \quad \underline{\underline{A}} = \underbrace{\frac{1}{2}\left(\underline{\underline{A}} + \underline{\underline{A}}^T\right)}_{\text{Symmetric Tensor}} + \underbrace{\frac{1}{2}\left(\underline{\underline{A}} - \underline{\underline{A}}^T\right)}_{\text{Skew-Symmetric Tensor}}$$

Here, $\underline{\underline{A}}^T$ is the transpose of the tensor $\underline{\underline{A}}$. $\underline{\underline{A}}^T$ has components that form a matrix whose columns are the rows of the matrix of components of $\underline{\underline{A}}$.

There is a special tensor that leaves a vector undisturbed. It is called the identity or unit tensor $\underline{\underline{I}}$.

$$\underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \underline{\underline{I}} \cdot \underline{x} = \underline{x} \text{ for any } \underline{x}$$

In index notation, we write $\underline{\underline{I}}$ as δ_{ij}, the Kronecker delta.

Symmetric tensors have a very special property. Remember that we define a tensor as the representation of some field at a point; at that point, there are **three special directions, orthogonal to each other**, associated with a symmetric tensor. When the tensor operates on a vector in one of these directions, it returns another vector pointing in the same (or exactly opposite) direction! The new vector, however, will have a different length in general. This multiplication factor in the length is called the **principal value** or **eigenvalue** of the tensor. If the eigenvalue is negative, the output vector from the operation will point opposite to the input vector.

Because there are, in general, three directions that are special, there are usually three distinct principal values, one associated with each direction. Even for tensors that are not symmetric, there are three principal values; however these need not all be real. Sometimes, two are complex. Even when the principal values are real, the directions associated with them need not be orthogonal if the tensor is not symmetric.

The problem for the principal or eigenvalues of $\underline{\underline{A}}$ is

$$\underline{\underline{A}} \cdot \underline{x} = \lambda \underline{x} \equiv \lambda \underline{\underline{I}} \cdot \underline{x}$$

Therefore,

$$\left[\underline{\underline{A}} - \lambda \underline{\underline{I}}\right] \cdot \underline{x} = \underline{0}$$

From linear algebra, for non-trivial solutions of the above system to exist, we must have

$$\det\left[\underline{\underline{A}} - \lambda \underline{\underline{I}}\right] = 0$$

The resulting third degree equation for the eigenvalues is

$$-\lambda^3 + I_A \lambda^2 - II_A \lambda + III_A = 0$$

and has three roots $\lambda_1, \lambda_{2,}$ and λ_3. When these roots are each used, in turn, and we solve for $\underline{x}$, we obtain an **eigenvector** that is known only to within an arbitrary multiplicative constant. Commonly, the eigenvector is normalized so that it has unit length.

From the above, you can see that corresponding to a symmetric tensor, there is a special rectangular Cartesian set of basis vectors of unit length. If we choose this as the basis set, the tensor will have a simple diagonal form with the diagonal components being the eigenvalues.

If you're wondering what happens when two eigenvalues are identical, it is easy to show that any vector in the plane normal to the third eigenvector (corresponding to the third eigenvalue) is acceptable as an eigenvector. In other words, on that plane, the tensor operating on a vector in any direction will yield a vector in the same direction with a magnification factor corresponding to the repeated eigenvalue.

If all three eigenvalues are identical, then any direction in space will be acceptable as the direction of the eigenvectors. Such a tensor is called **isotropic** for this reason. $\underline{\underline{I}}$ is an isotropic tensor with eigenvalues equal to unity. Any scalar multiple of $\underline{\underline{I}}$ also is isotropic.

Vector Calculus

If we consider a scalar field such as temperature, we find the rate of change with distance in some direction, x, by calculating $\partial \mathrm{T}/\partial x$. How can we represent the rate of change in three-dimensional space without specifying a particular direction? We do this via the **gradient operator**. The entity $\nabla \mathrm{T}$ [we call it "grad T"] is a vector field. In index notation we write the gradient operator as $\partial / \partial x_i$ where i is a free index, so that $\nabla \mathrm{T} = \partial T / \partial x_i$.

At a given point in space, the vector $\nabla \mathrm{T}$ points in the direction of greatest change of T. To obtain the rate of change of T at that point in any specified direction, $\underline{n}$, we simply "project" $\nabla \mathrm{T}$ in that direction.

$$\frac{\partial \mathrm{T}}{\partial n} = \nabla \mathrm{T} \cdot \underline{n}$$

unit vector

Surfaces in space on which a field has the same value everywhere are level surfaces. In the case of temperature fields, these surfaces are called **isotherms**. Along such a surface, the temperature cannot change. Therefore, the $\nabla \mathrm{T}$ vector is everywhere **normal** to isothermal surfaces since it must yield a value of zero when projected onto such surfaces.

It is straightforward to establish from the definition that

$$\nabla \equiv \underline{e}_{(1)} \frac{\partial}{\partial x_1} + \underline{e}_{(2)} \frac{\partial}{\partial x_2} + \underline{e}_{(3)} \frac{\partial}{\partial x_3}$$

in a rectangular Cartesian coordinate system (x_1, x_2, x_3).

Note that ∇ is an operator and not a vector. So, you should exercise care in manipulating it.

The ∇ operator is the generalization of a derivative. We can differentiate vector fields in more than one way.

Divergence

$\nabla \cdot \underline{v}$ or div $\underline{v}$ is called the divergence of the vector field $\underline{v}$. If the rectangular Cartesian components of $\underline{v}$ are $v_{1,} v_2, v_3,$ then

$$\nabla \cdot \underline{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{\partial v_i}{\partial x_i} \text{ in index notation}$$

As you can see, the result is a scalar field.

Curl

$\nabla \times \underline{v}$ or curl $\underline{v}$ is a vector field. As the name implies, it measures the "rotation" of the vector $\underline{v}$. Again, in $(x_{1,} x_2, x_3)$ coordinates,

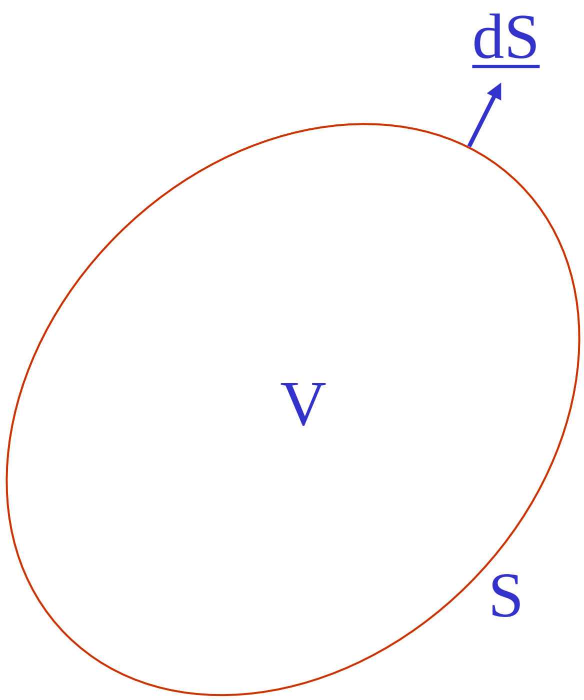
$$\nabla \times \underline{v} = \begin{vmatrix} \underline{e}_{(1)} & \underline{e}_{(2)} & \underline{e}_{(3)} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} = \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j} \text{ in index notation.}$$

There are two important theorems you should know. They are simply stated here without proof.

Divergence Theorem

If the volume V in space is bounded by the surface S,

$$\int_V \nabla \cdot \underline{a}\, dV = \int_S \underline{dS} \cdot \underline{a}$$



The vector field $\underline{a}$ should be continuous and differentiable. The symbol $\underline{dS}$ represents a vector surface element. If $\underline{n}$ is the unit normal to the surface,

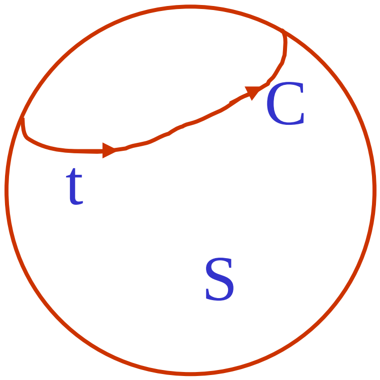
$$\underline{dS} = \underline{n}\, dS$$

The entity dV is a volume element. In the theorem, the left side is a volume integral and the right side is an integral over the surface that bounds the volume. Finally, $\underline{a}$ need not be a vector field, but can be a tensor field of any order.

The **divergence theorem**, also known as **Green's transformation,** is a very useful result that permits us to convert volume integrals into surface integrals. By applying it to an infinitesimal volume, you can visualize the physical significance of the divergence of a vector field at a point as the outward "flow" of the field from that point.

Stokes Theorem

This permits the conversion of integrals over a surface to those around a bounding curve. Imagine a surface that does not completely enclose a volume, but rather is open, such as a baseball cap. Let S be the surface and C, the curve that bounds it.



If a vector field $\underline{a}$ is defined everywhere necessary, and is continuous and differentiable, Stokes theorem states:

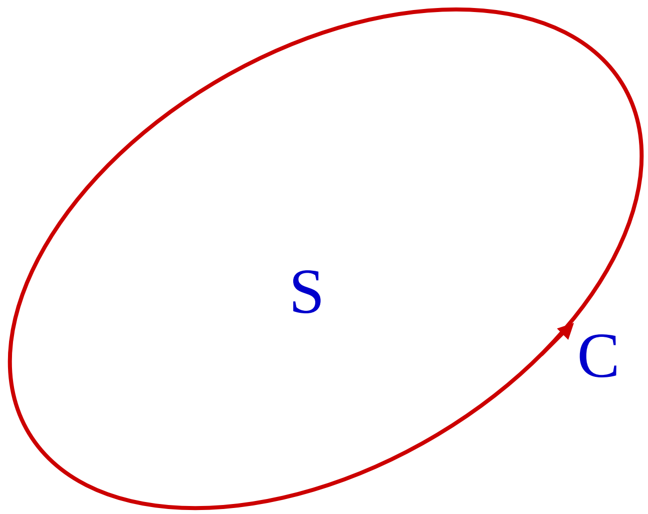
$$\int_S (\nabla \times \underline{a}) \cdot \underline{dS} = \int_C \underline{a} \cdot \underline{t}\, ds$$

$\underline{dS}$: vector area element on S

ds: scalar line element on C

$\underline{t}$: unit tangent vector on C

The integral on the right side is known as the **circulation** of $\underline{a}$ around the closed curve C. The field $\underline{a}$ appearing in the theorem can be replaced by a tensor field of any desired order.



By imagining the surface S to lie completely on the plane of the paper as shown, you can visualize the physical significance of $\nabla \times \underline{a}$. If you make S shrink to an infinitesimal area, the area integral on the left side becomes the product of the component of $\nabla \times \underline{a}$ normal to the plane of the paper and the area. The line integral is still the circulation around an infinitesimal closed loop surrounding the point. If $\underline{a}$ is the velocity field $\underline{v}$, by making the boundary an infinitesimal circle of radius ε, the right side can be seen to be approximately $2\pi\varepsilon v$, where v is the magnitude of the velocity around the loop. The left side is approximately $\pi\varepsilon^2 (\nabla \times v) \cdot \underline{n}$ where $\underline{n}$ is the unit normal to the plane of the paper. Therefore, $\frac{1}{2}(\nabla \times \underline{v}) \cdot \underline{n} \approx \frac{v}{\varepsilon}$, which becomes the instantaneous angular velocity of the fluid at the point on the plane of the paper as $\varepsilon \to 0$. Because there is nothing unique about the choice of the plane of the paper, we can see that $\frac{1}{2}(\nabla \times \underline{v})$ in fact represents the instantaneous angular velocity vector of a fluid element at a given point, the component of which in any direction is obtained by projecting in that direction.

The Gradient of a Vector Field

Just as we defined the gradient of a scalar field, it is possible to define the gradient of a vector or tensor field. If $\underline{v}$ is a vector field, $\nabla \underline{v}$ is a second-order tensor field. The rate of change of $\underline{v}$ in any direction $\underline{n}$ is given by

$$\frac{\partial \underline{v}}{\partial n} = \nabla \underline{v} \cdot \underline{n}$$