

Some Vector Operations in Curvilinear Coordinates

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Sometimes, we'll need the gradient or the Laplacian of a scalar field S or the divergence or curl of a vector field \mathbf{u} in basis sets associated with cylindrical or spherical polar coordinates. These can be found in Appendices in textbooks such as those by Batchelor or Bird, Stewart, and Lightfoot. I have reproduced the necessary results below.

Cylindrical Polar Coordinates (r, θ, z)

$$\nabla S = \frac{\partial S}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial S}{\partial \theta} \mathbf{e}_\theta + \frac{\partial S}{\partial z} \mathbf{e}_z$$

$$\nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\begin{aligned} \nabla \times \mathbf{u} = & \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \mathbf{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \mathbf{e}_\theta \\ & + \frac{1}{r} \left(\frac{\partial}{\partial r} [ru_\theta] - \frac{\partial u_r}{\partial \theta} \right) \mathbf{e}_z \end{aligned}$$

$$\nabla^2 S = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial S}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 S}{\partial \theta^2} + \frac{\partial^2 S}{\partial z^2}$$

Spherical Polar Coordinates (r, θ, ϕ)

$$\nabla S = \frac{\partial S}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial S}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial S}{\partial \phi} \mathbf{e}_\phi$$

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (u_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi}$$

$$\begin{aligned} \nabla \times \mathbf{u} = & \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} [u_\phi \sin \theta] - \frac{\partial u_\theta}{\partial \phi} \right) \mathbf{e}_r \\ & + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial u_r}{\partial \phi} - \frac{\partial}{\partial r} [ru_\phi] \right) \mathbf{e}_\theta \\ & + \frac{1}{r} \left(\frac{\partial}{\partial r} [ru_\theta] - \frac{\partial u_r}{\partial \theta} \right) \mathbf{e}_\phi \end{aligned}$$

$$\nabla^2 S = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial S}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial S}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 S}{\partial \phi^2}$$