Jacobian Determinant

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The geometrical significance of the Jacobian determinant is outlined here. Consider a transformation of a single rectangular Cartesian coordinate x to a new coordinate ξ. The line element dx is transformed to the new coordinate via

$$dx = \frac{dx}{d\xi} d\xi$$

In this case, the Jacobian determinant is simply the derivative $\frac{dx}{d\xi}$.

Now, consider an area element $dx\,dy$. For convenience in later generalization, we label the coordinates (x_1, x_2). Therefore, $x_1 = x$, and $x_2 = y$. Let us make a transformation to a new set of coordinates (ξ_1, ξ_2). The area element transforms as follows.

$$dx_1\,dx_2 = \left| \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2} - \frac{\partial x_2}{\partial \xi_1}\frac{\partial x_1}{\partial \xi_2} \right| d\xi_1\,d\xi_2$$

How did we obtain the above result? First, consider a differential change in the new variable ξ_1 while keeping the variable ξ_2 fixed. The components of the infinitesimal vector resulting from this change are

$$\left(\frac{\partial x_1}{\partial \xi_1} d\xi_1,\ \frac{\partial x_2}{\partial \xi_1} d\xi_1 \right)$$

In a like manner, we can write the components of the vector obtained by making a differential change in the second variable ξ_2 while keeping the variable ξ_1 fixed.

$$\left(\frac{\partial x_1}{\partial \xi_2} d\xi_2, \frac{\partial x_2}{\partial \xi_2} d\xi_2 \right)$$

These two vectors need not be orthogonal in general. Therefore, we need a result for the area of a parallelogram whose sides are the differential vectors written above. We know that this is the magnitude of the vector product (cross product) of the two vectors. This is the result given above for the area element.

Now, try evaluating this in the case of cylindrical polar coordinates (r,θ) with which you are familiar. Make a sketch on a piece of paper showing the coordinates (x,y) and measuring the angle θ counterclockwise from the x-direction.

Let $\xi_1 = r$ and $\xi_2 = \theta$. Therefore,

$$x_1 = x = r\cos\theta = \xi_1 \cos\xi_2$$
$$x_2 = y = r\sin\theta = \xi_1 \sin\xi_2$$

Then the vector obtained by making a differential change $d\xi_1 = dr$ has components $(dr\cos\theta, dr\sin\theta)$, where I have placed dr in front of each term for clarity. Similarly, the vector obtained by a differential change in the polar angle, which is $d\xi_2 = d\theta$, has components $(-r\sin\theta\, d\theta, r\cos\theta\, d\theta)$. From your sketch, verify that indeed these are the correct components.

The extension to three dimensions is straightforward. We make differential displacements in the three coordinates ξ_1, ξ_2 and ξ_3. We write the components of the infinitesimal vectors so obtained as

$$\left(\frac{\partial x_1}{\partial \xi_1} d\xi_1, \frac{\partial x_2}{\partial \xi_1} d\xi_1, \frac{\partial x_3}{\partial \xi_1} d\xi_1\right)$$
$$\left(\frac{\partial x_1}{\partial \xi_2} d\xi_2, \frac{\partial x_2}{\partial \xi_2} d\xi_2, \frac{\partial x_3}{\partial \xi_2} d\xi_2\right)$$
$$\left(\frac{\partial x_1}{\partial \xi_3} d\xi_3, \frac{\partial x_2}{\partial \xi_3} d\xi_3, \frac{\partial x_3}{\partial \xi_3} d\xi_3\right)$$

The volume of the parallelepiped formed with these three vectors as its sides is given by the magnitude of the triple scalar product, which is the absolute value of the determinant formed by the components of the vectors. From this, by taking out the common factor $d\xi_1\, d\xi_2\, d\xi_3$, we find that the volume dV is given by

$$dV = dx_1\, dx_2\, dx_3 = J(\xi_1, \xi_2, \xi_3)\, d\xi_1\, d\xi_2\, d\xi_3 = \left\| \begin{matrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{matrix} \right\| \, d\xi_1\, d\xi_2\, d\xi_3$$

where the inner set of vertical lines stands for the determinant, and the outer set is needed to yield the absolute value.