Solution of Partial Differential Equations
Combination of Variables

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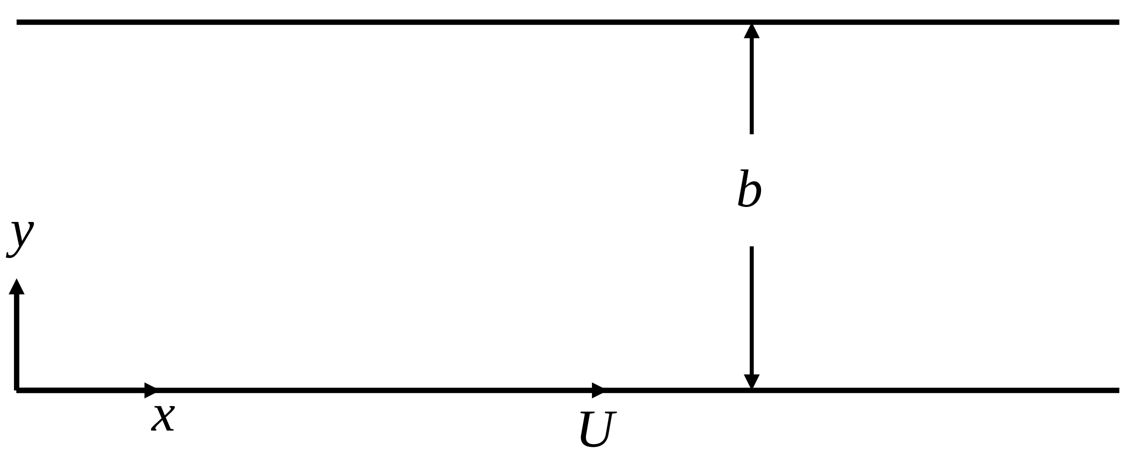
Introduction and Problem Statement

We encounter partial differential equations routinely in transport phenomena. Some examples are unsteady flow in a channel, steady heat transfer to a fluid flowing through a pipe, and mass transport to a falling liquid film. Here, we shall learn a method for solving partial differential equations that complements the technique of separation of variables. We shall also learn when the method can be used. We consider the same model problem, namely the motion induced in fluid contained between two long and wide parallel plates placed with a distance b between them as shown in the sketch below.



The fluid is initially assumed to be at rest. Motion is initiated by suddenly moving the bottom plate at a constant velocity of magnitude U in the $x-$direction. The velocity of the bottom plate is maintained at that value for all future values of time t while the top plate is held fixed in place. There is no applied pressure gradient, with motion being caused strictly by the movement of the bottom plate.

We shall assume the flow to be incompressible with a constant density ρ and Newtonian with a constant viscosity μ. We neglect edge effects in the $z-$direction so that we can set $v_z = 0$ and $\dfrac{\partial \boldsymbol{v}}{\partial z} = 0$, and assume fully developed flow, implying $\dfrac{\partial \boldsymbol{v}}{\partial x} = 0$. Here, $\boldsymbol{v}$ stands for the velocity vector, and the subscripts denote components.

It can be established from the continuity equation and the kinematic condition at one of the walls that $v_y = 0$. Therefore, the only non-zero velocity component is $v_x\left(t, y\right)$, which can be shown to satisfy the following partial differential equation.

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2} \tag{1}$$

In Equation (1), t represents time, and ν is the kinematic viscosity. The initial condition is

$$v_x\left(0, y\right) = 0 \tag{2}$$

and the boundary conditions are

$$v_x\left(t, 0\right) = U \tag{3}$$

and

$$v_x\left(t, b\right) = 0 \tag{4}$$

Using separation of variables, we obtained a solution of these equations that can be written as follows.

$$\frac{v_x\left(t, y\right)}{U} = 1 - \frac{y}{b} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\exp\left[-n^2 \pi^2 \frac{\nu t}{b^2}\right]}{n} \sin\left(\frac{n\pi y}{b}\right) \tag{5}$$

The infinite series in Equation (5) is uniformly convergent for all values of time t. The exponential factor plays a strong role in assuring that the terms decrease rapidly with increasing values of n so that only a few terms are necessary to calculate an accurate value of the velocity at moderate to large values of time, corresponding to the scaled time $\nu t / b^2$ not being too small compared to unity. But, if we attempt to calculate the sum numerically for small values of time ($\nu t / b^2$ small compared with unity) when the exponential factor is not as helpful, we find that a large number of terms needs to be included to obtain a sufficiently accurate answer. Therefore, in this module we seek a solution technique that will permit us to calculate the velocity field accurately without too much labor for small values of time.

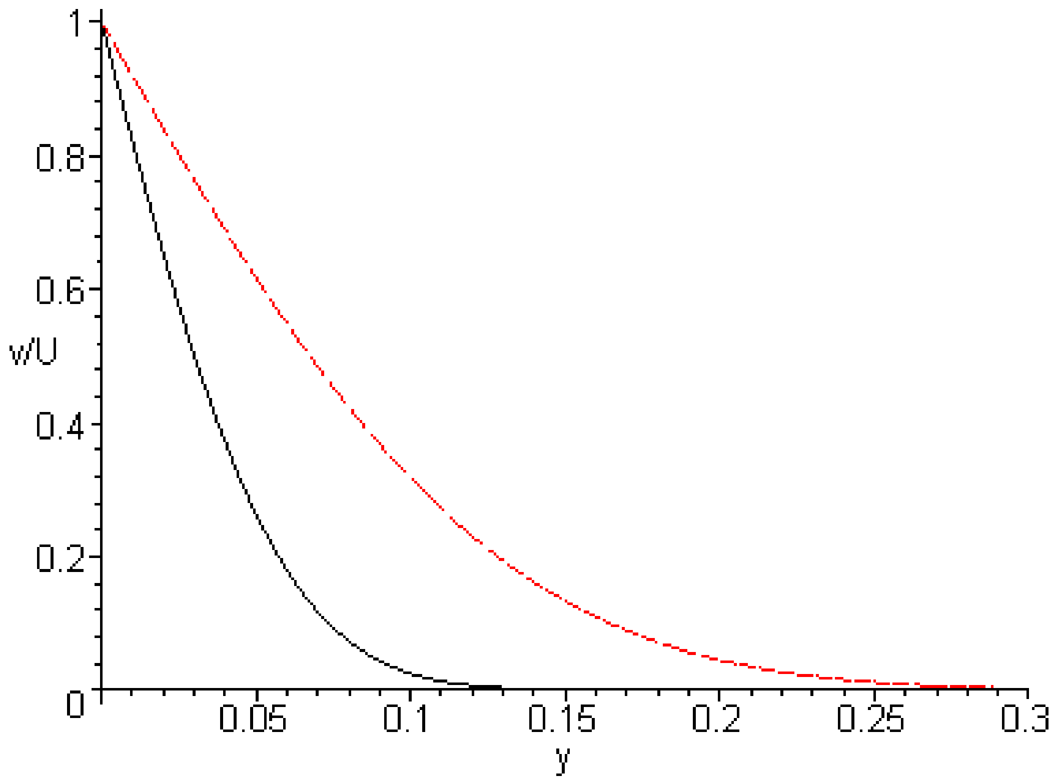
Physically, at values of time t for which the scaled time $\nu t / b^2$ is small compared to unity, the effect of the motion of the bottom plate is only felt by the fluid up to a small distance (depth of penetration) from the moving plate. Outside of this region of influence, the fluid is practically stationary. Therefore, one might approximate the system for such small values of time by another in which the top plate is absent. This problem was first considered by Lord Rayleigh, and therefore is known as Rayleigh's problem. Mathematically, we replace the boundary condition at the top plate, given in Equation (4), with

$$v_x(t,\infty)=0 \tag{6}$$

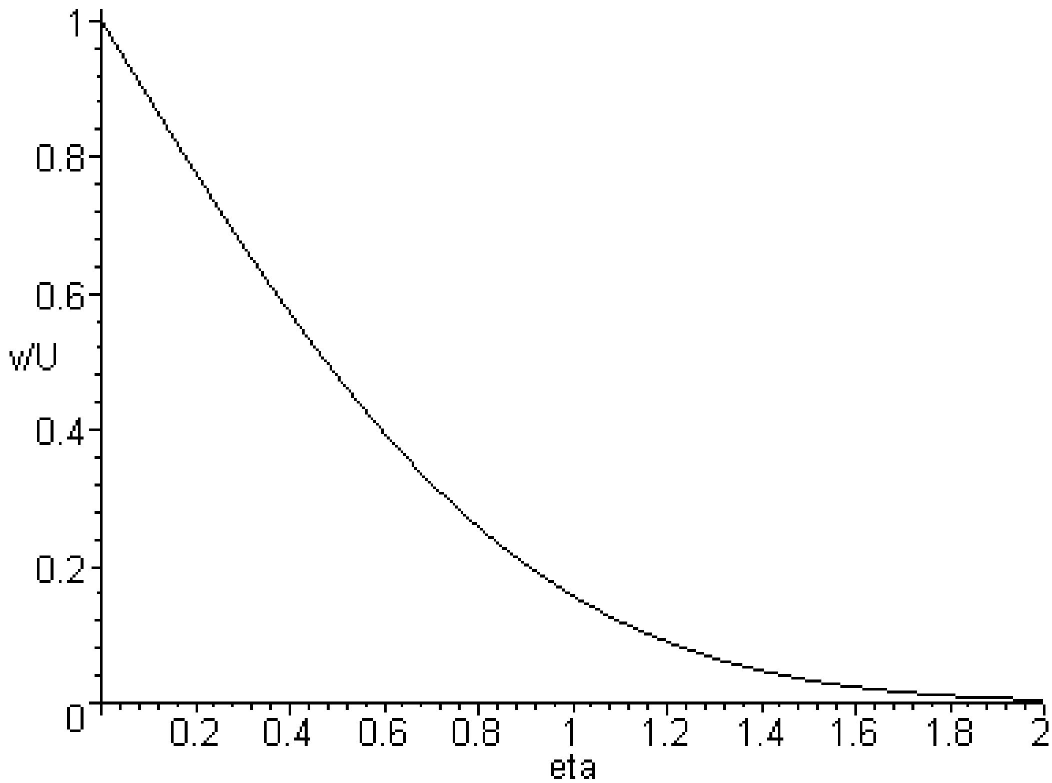
Note that to be precise, we must write Equation (6) as $v_x(t, y\to\infty)\to 0$, and the equation must be read to imply only such a meaning.

A Speculation

There is neither a natural length scale in the problem, nor a natural time scale. We can use the reference velocity U as a natural scale for the velocity v_x, but it is convenient to work with the remaining physical variables just as they are. The solution of Equations (1) to (3) and (6) is qualitatively sketched below for two different values of time. In the figure, the symbol v is used to represent v_x.



The solid (black) line corresponds to a small value of time, and the dashed (red) line to a larger value of time. We can see how the change in velocity made at the bottom plate at time zero propagates deeper into the fluid with increasing time. It is tempting to speculate that these profiles are similar in shape. By implying similarity of shape, we mean that scaling the distance variable with the thickness of the affected region $\delta(t)$ should lead to these two curves and others like them collapsing into a single universal curve. In mathematical language, if we define a certain combination of the original variables as a new variable $\eta = y/\delta(t)$, can we expect the velocity field $v_x(t,y)$ to become a function $U\phi(\eta)$ that depends only on the single new variable? This speculation is shown in the sketch below.



The transformation to η is known as a "similarity transformation" and the variable η is termed a "similarity variable."

Solution by Combination of Variables

We now proceed to state the above speculation in mathematical form and follow through the consequences. This is the method of "Combination of Variables."

Assume

$$v_x(t,y) = U\phi(\eta) \tag{7}$$

where

$$\eta = \frac{y}{\delta(t)} \tag{8}$$

and $\delta(t)$ is a function that is yet to be determined. Note that we always can transform from two independent variables to two new independent variables, but to transform to a single new variable is not always possible. Therefore, we need to insert Equations (7) and (8) into the governing equation and the initial and boundary conditions and see if the process leads to a consistent mathematical framework. For this purpose, we shall use the chain rule of differentiation as needed.

$$\frac{\partial v_x}{\partial t} = U\,\frac{\partial \eta}{\partial t}\frac{d\phi}{d\eta} = -U\,\frac{y}{\delta^2}\frac{d\delta}{dt}\frac{d\phi}{d\eta} = -U\,\frac{\eta}{\delta}\frac{d\delta}{dt}\frac{d\phi}{d\eta} \tag{9}$$

Note that when writing the derivative of ϕ with respect to η, we already have assumed that ϕ can depend explicitly only on the single variable η and used the ordinary derivative instead of the partial derivative. If our conjecture proves to be incorrect, and ϕ were to depend explicitly on both η and t, the above chain rule result will need to be modified to include a partial derivative of ϕ with respect to time.

Let us now obtain expressions for the derivatives with respect to y.

$$\frac{\partial v_x}{\partial y} = U\,\frac{\partial \eta}{\partial y}\frac{d\phi}{d\eta} = \frac{U}{\delta}\frac{d\phi}{d\eta} \tag{10}$$

and

$$\begin{aligned}\frac{\partial^2 v_x}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{U}{\delta}\frac{d\phi}{d\eta}\right) = \frac{U}{\delta}\frac{\partial}{\partial y}\left(\frac{d\phi}{d\eta}\right) \\ &= \frac{U}{\delta}\frac{\partial \eta}{\partial y}\frac{d}{d\eta}\left(\frac{d\phi}{d\eta}\right) = \frac{U}{\delta^2}\frac{d^2\phi}{d\eta^2}\end{aligned} \tag{11}$$

Substituting Equations (9) and (11) into the governing differential equation for v_x (Equation (1), leads to the following equation after slight rearrangement.

$$\phi'' + \eta\left[\frac{\delta\delta'}{\nu}\right]\phi' = 0 \tag{12}$$

In writing Equation (12), we have used the expedient of designating derivatives with respect to the argument of each function with primes. Recall that we assumed that ϕ explicitly depends only on the similarity variable η. But, Equation (12) suggests that time also will explicitly appear in the result for ϕ because of the presence of the time-dependent term $\delta\delta'$. We have not yet specified $\delta(t)$, however. Here is our chance to do so and eliminate the inconsistency at the same time. We choose

$$\delta\delta' = C\nu \tag{13}$$

where C is an arbitrary constant. Later, we shall see that the value of C will affect the result for $\delta(t)$, but will not affect the final solution for $v_x(t,y)$. Therefore, for convenience, we set $C = 2$, writing Equation (12) as

$$\phi'' + 2\eta\phi' = 0 \tag{14}$$

We now need to transform the initial and boundary conditions. Note that there are three conditions on the velocity field $v_x(t, y)$, but only a second order differential equation for $\phi(\eta)$. The specification of the arbitrary constants that arise in the integration of the latter requires only two conditions.

First, consider the boundary condition at the bottom surface $y = 0$, given in Equation (3). This transforms in a straightforward manner to

$$\phi(0) = 1 \tag{15}$$

The fact that a quiescent condition is approached as $y \to \infty$, described by Equation (6), becomes

$$\phi(\infty) = 0 \tag{16}$$

The initial condition, given in Equation (2), transforms to

$$\phi\left(\frac{y}{\delta(0)}\right) = 0 \tag{17}$$

and we see that we have not completely eliminated the original variables from appearing explicitly in the problem statement for ϕ. To remove this inconsistency, and at the same time select an initial condition for $\delta(t)$, we must set

$$\delta(0) = 0 \tag{18}$$

The choice in Equation (18) makes Equation (17) collapse into Equation (16); therefore, the three conditions on $v_x(t, y)$ yield two conditions on $\phi(\eta)$ and one initial condition on $\delta(t)$, and we have a completely consistent mathematical framework for the problems for $\phi(\eta)$ and $\delta(t)$. Note that by this approach of "Combination of Variables" we have reduced the solution of the original partial differential equation to that of two ordinary differential equations for these two functions.

First, the general solution of Equation (14) can be written as

$$\phi(\eta) = a_1 + a_2 \int_0^{\eta} e^{-\gamma^2} d\gamma \tag{19}$$

where a_1 and a_2 are constants of integration that must be determined by applying the boundary conditions given in Equations (15) and (16). Use of these conditions leads to the result

$$\phi(\eta) = \text{erfc}(\eta) \tag{20}$$

where " erfc " means "complementary error function." This function is defined as follows.

$$\text{erfc}(\eta) = 1 - \text{erf}(\eta) \tag{21}$$

where the "error function" " erf " is defined as

$$erf(\eta) = \frac{\int_0^{\eta} e^{-\gamma^2} d\gamma}{\int_0^{\infty} e^{-\gamma^2} d\gamma} = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\gamma^2} d\gamma \tag{22}$$

You can find out more about the error function and the complementary error function from Abramowitz and Stegun [1].

The solution of Equation (13) with the constant $C = 2$, when specialized using the initial condition given in Equation (18), is

$$\delta(t) = 2\sqrt{\nu t} \tag{23}$$

When this result for $\delta(t)$ is used in Equation (8) in which η is defined, the solution for the velocity field can be written as

$$v_x(t, y) = U \,\text{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right) \tag{24}$$

If we had made a different choice of value for the constant C that appears in Equation (13), it would have affected the results as follows.

$$\delta(t) = \sqrt{2C\nu t} = \sqrt{\frac{C}{2}}\, 2\sqrt{\nu t} \tag{25}$$

$$\phi(\eta) = erfc\left(\sqrt{\frac{C}{2}}\,\eta\right) \tag{26}$$

You can see that when the definition of η given in Equation (8) is used in Equation (26), along with $\delta(t)$ from Equation (25), the factor $\sqrt{C/2}$ cancels out, leading to the same result for the velocity field given in Equation (24). You may wonder about the uncertainty in the value of $\delta(t)$, which is the thickness of the "affected region," caused by the indeterminacy of the value of C. This is perfectly natural because in a diffusive process, the influence of a change is felt everywhere in the fluid instantaneously. This means that there can be no unambiguous definition of a finite thickness for the affected region; only its scaling can be established uniquely. The complementary error function assumes a value of 4.678×10^{-3} when its argument is 2. Therefore, at a distance $y = 4\sqrt{\nu t}$, the velocity would be less than 0.5% of the value at the surface of the moving plate, and can be considered negligible for practical purposes. Because of this, the estimate $\left(4\sqrt{\nu t}\right)$ is sometimes used for the thickness of the region influenced by the sudden movement initiated at the boundary.

Summary

In this module, we have learned the method of combination of variables for solving partial differential equations; it complements the method of separation of variables. First, we identified the governing partial differential equation and boundary conditions for our system. Then we

1. noted that the effect of a boundary condition imposed at time zero is felt in a region near that boundary that is small in extent for small values of time and used this fact to replace the boundary condition at the other boundary with one at infinity;

2. assumed that the dependence of the velocity field on the two independent variables can be expressed as a dependence on a single new similarity variable;

3. traced the consequences of this similarity hypothesis mathematically, requiring that the original independent variables not be allowed to appear explicitly in the problem posed in the new similarity variable;

4. obtained an ordinary differential equation for the thickness of the affected region and another ordinary differential equation for the velocity field;

5. collapsed the three boundary conditions on the velocity field into two on the velocity field as expressed in the similarity variable, also yielding an initial condition for the thickness of the affected region;

6. solved these ordinary differential equations to obtain results for the thickness of the affected region and the velocity field;

7. noted that the thickness of the affected region can only be defined to within a multiplicative arbitrary constant, whereas the velocity field is uniquely determined.

The important features of the method are that the domain must be semi-infinite, and the boundary condition at infinity must be the same as the initial condition; even though the problem we posed is linear, the method is equally applicable to non-linear problems.

Concluding Remarks

The problem of unsteady one-dimensional heat conduction in a semi-infinite solid slab (or a quiescent liquid layer) in the $y-$direction, when the temperature at the surface $y = 0$ is changed to a new value at time zero, is described by the same governing equations and boundary conditions. The assumptions are that there are no sources or sinks, heat transport occurs only by conduction with a constant thermal conductivity, the density and specific heat of the material are constant, and that the slab is very long and very wide so that end effects and edge effects can be neglected. By analogy, it can be seen that the same equations also describe unsteady diffusion in a similar situation. All of these cases can be handled by the same solution method. Note that unlike separation of variables, combination of variables does not require the system of governing equation and boundary conditions to be linear. This method has used successfully in solving the Navier-Stokes equations including inertia (and therefore non-linear) in forced boundary layer flows, and also in solving problems of natural convection in boundary layers wherein the fluid mechanics and heat transport problems lead to coupled non-linear governing equations.

Reference

1. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965.