An Introduction to Asymptotic Expansions

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Asymptotic expansions are used in analysis to describe the behavior of a function in a limiting situation. When a function $y(x,\varepsilon)$ depends on a small parameter ε, and the solution of the governing equation for this function is known when $\varepsilon = 0$, a perturbation method may prove useful in obtaining a solution for small values of ε. Such an approach is particularly attractive when the governing equation is nonlinear and no general techniques are available for exact solution. If ε appears as a multiplicative factor in a term in the governing equation, the standard approach is to try a power series solution of the following form:

$$y(x,\varepsilon) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \ldots \qquad (1)$$

where the symbol … stands for higher order terms. The series is inserted into the governing equation and boundary conditions, and coefficients of like powers of ε are then grouped to obtain a series of equations for the coefficient functions $y_j(x)$, which are then solved in a sequential manner. The resulting series need not converge for any value of ε; nevertheless, the solution can be useful in approximating the function $y(x,\varepsilon)$ when ε is small.

Convergent and Asymptotic Series

Computationally, a convergent series $\sum_{j=1}^{\infty} a_j$ is not always useful, because convergence is a concept relating to the behavior of the terms in the series at the tail end, that is, as $j \to \infty$. That a series converges says nothing about how rapidly the terms will decrease in magnitude. On the other hand, in an asymptotic series, the terms will usually decrease rapidly with j at first for sufficiently small ε. Sometimes, they may begin to increase with increasing j at some point after decreasing initially. When the terms are decreasing rapidly, if we sum just the first few terms and we know that the error incurred is of the order of the next term, we can get a good estimate of the sum. This is why asymptotic series, even when divergent, are practically useful. The main problem with asymptotic series is that one never knows how accurate the answer is. The results must be validated by comparison with some other representation of the expected answer. Nevertheless, asymptotic series may be the only means of obtaining an analytical solution of a difficult problem, and are used commonly for this purpose.

To illustrate the ideas regarding computational utility, write a computer program to sum the first 100 terms in the Taylor series for $\sin\theta$ given below, first for θ = 0.5, and then for θ = 10^4.

$$\sin\theta = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \qquad (2)$$

The series is known to be uniformly convergent for all values of θ. The result for $\theta = 10^4$ reflects the fact that the precision of machine computation is finite. Errors introduced by the limited precision lead to an absurd result when the sum is calculated. You can see a demonstration of this by printing each term and the sum as successive terms are added.

Now, consider the following series for the complementary error function.

$$erfc(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}}\left[1+\sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{\left(2x^2\right)^n}\right] \tag{3}$$

This series diverges for all values of x. It is an asymptotic series that represents the function in the limit as $x \to \infty$. In spite of its divergence, it is useful for computing the complementary error function for large values of x, because the terms in the series decrease rapidly with increasing n for small values of n and the error incurred by truncating the series at a certain term is of the order of magnitude of the next term, which is much smaller than the term retained as long as x is large and we use only a small number of terms. Try calculating results from this series for $x = 5$, 10, and 20 and check the sum after adding each term against the exact result. Also, see if you can demonstrate to yourself that this is a divergent series.

Some Basic Concepts

Some basic concepts in using asymptotic series are described next. Two symbols are commonly used to describe the behavior of a function $f(\varepsilon)$ in the limit as $\varepsilon \to 0$. They are "O" and "o"and are termed big "oh" and little "oh." If we have two different functions of ε, namely $f(\varepsilon)$ and $g(\varepsilon)$, we say that

$$f(\varepsilon) = O\left(g(\varepsilon)\right) \quad \text{if} \quad \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} < \infty \tag{4}$$

In words, this is stated as follows: $f(\varepsilon)$ is of the order of $g(\varepsilon)$.

If the limit is zero, then the symbol o is used.

$$f(\varepsilon) = o\left(g(\varepsilon)\right) \quad \text{if} \quad \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 0 \tag{5}$$

In the above, the function $g(\varepsilon)$ is termed a "gauge function." A common set of gauge functions is the set of powers of epsilon $\left(1, \varepsilon, \varepsilon^2, ...\right)$. These powers are often used to describe the behavior of some other function of epsilon. For example, we may write

$$\sin\varepsilon \sim \varepsilon \tag{6}$$

which should be read as "sine epsilon is asymptotically equal to epsilon." Even though the phrase "as ε approaches zero" is omitted, it is implied. Of course,

$$\tan \varepsilon \sim \varepsilon \tag{7}$$

at leading order, so that we see that different functions can have identical asymptotic representations.

Power series are just one type of asymptotic series. A more general asymptotic series for a function $y(x,\varepsilon)$ is of the form

$$y(x,\varepsilon) = \sum_{n=0}^{N} f_n(\varepsilon)\, y_n(\varepsilon) \tag{8}$$

Note that we have terminated the series at a finite value of the index. Therefore, convergence is not an issue here. The functions $f_n(\varepsilon)$ must satisfy

$$\lim_{\varepsilon \to 0} \frac{f_{n+1}(\varepsilon)}{f_n(\varepsilon)} = 0, \qquad n = 0,1,2,... \tag{9}$$

This means that each member of the set of functions approaches zero more rapidly than the previous member as $\varepsilon \to 0$. We call the set of functions $\{f_n(\varepsilon)\}$ an asymptotic sequence if the members satisfy the condition given in Equation (9). Note that the set of powers of epsilon is indeed an asymptotic sequence.

The coefficient functions $y_n(x)$ can be determined uniquely from the property of the members of an asymptotic sequence noted above. First, by dividing both sides of Equation (8) by $f_0(\varepsilon)$ and taking the limit as $\varepsilon \to 0$, we obtain the following result for the leading order coefficient $y_0(x)$.

$$y_0(x) = \lim_{\varepsilon \to 0} \frac{y(x,\varepsilon)}{f_0(\varepsilon)} \tag{10}$$

Now, subtract $f_0(\varepsilon)\, y_0(x)$ from both sides of Equation (8), divide by $f_1(\varepsilon)$, and take the limit as $\varepsilon \to 0$. This yields

$$y_1(x) = \lim_{\varepsilon \to 0} \frac{y(x,\varepsilon) - f_0(\varepsilon)\, y_0(\varepsilon)}{f_1(\varepsilon)} \tag{11}$$

Using this procedure, it is straightforward to show that the coefficient function $y_j(x)$ in the asymptotic series can be written as

$$y_j(x) = \lim_{\varepsilon \to 0} \frac{y(x,\varepsilon) - \sum_{n=0}^{j-1} f_n(\varepsilon)\, y_n(\varepsilon)}{f_j(\varepsilon)}, \qquad j = 1, 2, 3, \ldots \tag{12}$$

The coefficients in the asymptotic series for a given function depend on the choice of the sequence; once the sequence is defined, the coefficients are uniquely determined by Equations (10) and (12). In a given problem, we usually do not know the dependence of $y(x,\varepsilon)$ on ε so that the results in these equations should be regarded only as formal definitions of the coefficient functions. Next, we demonstrate how these coefficients are determined in example cases.

An Integral

Consider the integral $I(\varepsilon)$ defined as shown below.

$$I(\varepsilon) = \int_0^\infty \frac{e^{-t}}{1+\varepsilon t}\, dt \tag{13}$$

Proceeding to integrate by parts, we obtain

$$\begin{aligned} I(\varepsilon) &= \left[\frac{1}{1+\varepsilon t} \left\{ -e^{-t} \right\} \right]_0^\infty - \int_0^\infty \left\{ -e^{-t} \right\} \left\{ -\frac{\varepsilon}{(1+\varepsilon t)^2} \right\} dt \\ &= 1 - \varepsilon \int_0^\infty \frac{e^{-t}}{(1+\varepsilon t)^2}\, dt \\ &= 1 - \varepsilon \left(\left[\frac{1}{(1+\varepsilon t)^2} \left\{ -e^{-t} \right\} \right]_0^\infty - \int_0^\infty \left\{ -e^{-t} \right\} \left\{ -\frac{2\varepsilon}{(1+\varepsilon t)^3} \right\} dt \right) \\ &= 1 - (1!)\varepsilon + (2!)\varepsilon^2 \int_0^\infty \frac{e^{-t}}{(1+\varepsilon t)^3}\, dt \end{aligned} \tag{14}$$

Continuing to integrate by parts in this manner, we can show that

$$\begin{aligned} I(\varepsilon) = 1 - (1!)\,\varepsilon + (2!)\,\varepsilon^2 - \ldots + (-1)^{n-1} \left([n-1]! \right) \varepsilon^{n-1} \\ + (-1)^n (n!)\, \varepsilon^n \int_0^\infty \frac{e^{-t}}{(1+\varepsilon t)^{n+1}}\, dt \end{aligned} \tag{15}$$

As an infinite series,

$$I(\varepsilon) = \sum_{n=0}^{\infty} (-1)^n (n!)\, \varepsilon^n \tag{16}$$

is divergent for all values of ε. But, for relatively small values of ε, the series in Equation (16), truncated after a small number of terms, provides a good approximation of the integral.

A Differential Equation

Consider the differential equation

$$y' + \varepsilon y = 0 \tag{17}$$

along with the initial condition

$$y(0) = 1 \tag{18}$$

We know that the solution is

$$y = e^{-\varepsilon x} \tag{19}$$

Let us see how an asymptotic expansion can be developed for $y(x, \varepsilon)$. Write

$$y = \sum_{k=0}^{\infty} \varepsilon^k y_k(x) \tag{20}$$

Substitute this expansion into the governing equation (17), yielding

$$\sum_{k=0}^{\infty} \varepsilon^k y_k' + \sum_{k=0}^{\infty} \varepsilon^{k+1} y_k = 0 \tag{21}$$

Rearrange this equation to write it as

$$\sum_{k=0}^{\infty} \varepsilon^k \left(y_k' + y_{k-1} \right) = 0 \tag{22}$$

with the convention that $y_{-1} = 0$. We can see that by taking the limit $\varepsilon \to 0$, we obtain

$$y_0' = 0 \tag{23}$$

and by subtracting this result from Equation (22), dividing both sides by ε, and taking the limit $\varepsilon \to 0$ again, we get

$$y_1' = -y_0 \tag{24}$$

Repeating the process as many times as needed leads to

$$y_k' = -y_{k-1}, \quad k = 0, 1, 2, \ldots \tag{25}$$

We could also have written Equation (25) by formally setting the coefficient of ε^k to zero in Equation (22) for each value of k.

By inserting the asymptotic expansion given in Equation (20) into the initial condition, we obtain

$$\sum_{k=0}^{\infty} \varepsilon^k y_k(0) = 1 \tag{26}$$

which yields

$$y_k(0) = \delta_{k0} \tag{27}$$

where $\delta_{ij} = 1$ when $i = j$ and 0 otherwise. It is known as the Kronecker delta.

The solution of $y_0' = 0$ along with $y_0(0) = 1$ is $y_0 = 1$. Using this, we can solve

the equation for y_1, which is $y_1' = -y_0 = -1$, along with $y_1(0) = 0$ to yield $y_1 = -x$. By continuing the process, we find $y_2(x) = x^2/2!$, $y_3 = -x^3/3!$, and so on. The solution for $y(x,\varepsilon)$ can be written as

$$y(x,\varepsilon) = \sum_{k=0}^{\infty} (-1)^k \frac{(\varepsilon x)^k}{k!} \tag{28}$$

which is the Taylor series for the exponential function $y = e^{-\varepsilon x}$. This series happens to converge uniformly for all values of ε and x. In this example, our attempt to find a power series expansion in ε has led to a convergent series, even though we cannot expect the same in other problems.

Concluding Remarks

We have seen how a useful approximation to the solution of problems involving a small parameter can be obtained by expanding in an asymptotic series in that parameter. This method is known as "perturbation." It can be shown that the simple technique illustrated here fails if the small parameter multiplies the highest order derivative in a differential equation. This is because the order of the differential equation is reduced when the small parameter is set equal to zero. This leads to qualitative differences in the solution, and in boundary value problems, the inability to satisfy the complete set of boundary conditions on the problem. Also, a simple perturbation method can fail even when the small parameter only multiplies a low order derivative if the domain is unbounded, as can occur in idealized mathematical problems. These problems are handled by using "singular perturbation" techniques. You can learn more about perturbation methods from any of the following references.

References

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