Viscous Dissipation Term

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When the dot product of each term in the Navier-Stokes equation is taken with the velocity vector $\boldsymbol{u}$, the result can be cast in the form of an equation for the time rate of change of the kinetic energy of the fluid per unit volume. An important term that appears in the result for this quantity is the rate at which the work done against viscous forces is irreversibly converted into internal energy. This is known as "viscous dissipation." The viscous dissipation per unit volume is written as $\boldsymbol{\tau} : \nabla \boldsymbol{u} = \mu \Phi_{\boldsymbol{v}}$ where Φ_v for a Newtonian fluid is given below in different coordinate systems.

Rectangular Cartesian Coordinates (x, y, z)

$$\Phi_v = 2\left[\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right] + \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)^2$$

$$+ \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)^2 - \frac{2}{3}(\nabla \bullet \boldsymbol{u})^2$$

Cylindrical Polar Coordinates (r, θ, z)

$$\Phi_v = 2\left[\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right]$$

$$+ \left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]^2 + \left[\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right]^2$$

$$+ \left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right]^2 - \frac{2}{3}(\nabla \bullet \boldsymbol{u})^2$$

Spherical Polar Coordinates (r,θ,ϕ)

$$\Phi_v = 2\left[\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta\right)^2\right]$$

$$+\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]^2 + \left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{u_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}\right]^2$$

$$+\left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right]^2 - \frac{2}{3}\left(\nabla \bullet \boldsymbol{u}\right)^2$$

Reference

R.B. Bird, W.E. Stewart, and E.N. Lightfoot, Transport Phenomena, Wiley, 2007.