Some Vector Operations in Curvilinear Coordinates

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Sometimes, we'll need the gradient or the Laplacian of a scalar field S or the divergence or curl of a vector field $\boldsymbol{u}$ in basis sets associated with cylindrical or spherical polar coordinates. These can be found in Appendices in textbooks such as those by Batchelor or Bird, Stewart, and Lightfoot. I have reproduced the necessary results below.

Cylindrical Polar Coordinates (r,θ,z)

$$\nabla S = \frac{\partial S}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial S}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial S}{\partial z}\boldsymbol{e}_z$$

$$\nabla \bullet \boldsymbol{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(r u_r\right) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla \times \boldsymbol{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\boldsymbol{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\boldsymbol{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}\left[r u_\theta\right] - \frac{\partial u_r}{\partial \theta}\right)\boldsymbol{e}_z$$

$$\nabla^2 S = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 S}{\partial \theta^2} + \frac{\partial^2 S}{\partial z^2}$$

Spherical Polar Coordinates (r,θ,ϕ)

$$\nabla S = \frac{\partial S}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial S}{\partial \theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial S}{\partial \phi}\boldsymbol{e}_\phi$$

$$\nabla \bullet \boldsymbol{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(u_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}$$

$$\nabla \times \boldsymbol{u} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}\left[u_\phi \sin\theta\right] - \frac{\partial u_\theta}{\partial \varphi}\right)\boldsymbol{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{\partial}{\partial r}\left[r u_\phi\right]\right)\boldsymbol{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}\left[r u_\theta\right] - \frac{\partial u_r}{\partial \theta}\right)\boldsymbol{e}_\phi$$

$$\nabla^2 S = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial S}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial S}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 S}{\partial \phi^2}$$

References

1. G.K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, 1967.

2. R.B. Bird, W.E. Stewart, and E.N. Lightfoot, Transport Phenomena, Wiley, 2007.