Stokes Flow – Invariant Representation of Solutions

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The term "invariant representation" implies that the results are written in a form that is independent of the coordinate system or, equivalently, the basis set used for expressing the components of vectors.

In an early homework assignment, we learned about solutions of the Laplace equation written in such invariant notation involving only the position vector x_i and its length r. These are called "vector harmonics" even though the solutions are tensors, and are ordered such that at each level, the order increases by 1. There are two sets of such solutions. One is the set of decaying harmonics. The general decaying harmonic is given by

$$\phi_{-(n+1)} = \frac{(-1)^n}{1\cdot 3\cdot 5\cdots(2n-1)} \underbrace{\nabla\nabla\nabla\cdots\nabla}_{n \text{ times}}\left(\frac{1}{r}\right), \qquad n = 0, 1, 2, ...$$

The first few are written below.

$$\frac{1}{r}, \quad \frac{x_i}{r^3}, \; \frac{x_i x_j}{r^5} - \frac{\delta_{ij}}{3r^3}, \; \frac{x_i x_j x_k}{r^7} - \frac{x_i\delta_{jk} + x_j\delta_{ki} + x_k\delta_{ij}}{5r^5}$$

You can see that these solutions approach 0 as $r \to \infty$. Growing harmonics, in contrast, grow with distance from the origin. These are obtained from the decaying harmonics as $r^{2n+1}\phi_{-(n+1)}$. The first few growing harmonics are given below.

$$1, \; x_{i,} \;\; x_i x_j - \frac{r^2}{3}\delta_{ij}, \, x_i x_j x_k - \frac{r^2}{5}\left(x_i\delta_{jk} + x_j\delta_{ki} + x_k\delta_{ij}\right), \cdots$$

Absolute and Pseudotensors

We'll find these sets of harmonic functions very useful in constructing solutions of Stokes problems. The important idea in each case will be that we'll be looking for solutions of Laplace's equation in one case for a scalar field (pressure) and in the other for a vector field (homogeneous solution for the velocity). The pressure and velocity fields are entities that are called absolute tensors. Absolute tensors are unaffected by whether the coordinate system (and therefore the basis set) is right or left-handed. On the other hand, we are familiar with vectors and tensors that change sign if the basis set is changed from one that is right-handed to one that is left-handed. One example is the cross product of two absolute vectors. Another way to obtain a pseudovector is by taking the curl of an absolute vector field; an example is the vorticity vector field, defined as the curl of the velocity field. Performing a second such operation on a pseudovector such as taking a curl will yield an absolute entity. Also, if the cross product of a pseudovector and an absolute vector is formed, the result with be an absolute vector. You will find a more detailed discussion of the distinction between absolute and pseudotensors in page 525 of Leal (1).

Solution of the Stokes equation

Now, let us begin with the Stokes equation for an incompressible flow.

$$\mu \nabla^2 \boldsymbol{u} = \nabla p$$

Also, an incompressible velocity field must satisfy $\nabla \bullet \boldsymbol{u} = 0$. If we now take the divergence of the Stokes equation, $\nabla \bullet \left(\mu \nabla^2 \boldsymbol{u} - \nabla p \right) = 0$, which can be rearranged to $\mu \nabla^2 \left(\nabla \bullet \boldsymbol{u} \right) - \nabla \bullet \nabla p = 0$. The first term is zero because the flow is incompressible, leading to the result that $\nabla^2 p = 0$. Therefore, the pressure field in every Stokes problem must be a harmonic function, that is, a function which satisfies the Laplace equation.

We can construct a solution of the Stokes equation by adding a homogeneous solution and a particular solution. The homogeneous solution $\boldsymbol{u}^{(H)}$ must also satisfy Laplace's equation $\nabla^2 \boldsymbol{u}^{(H)} = \boldsymbol{0}$. The particular solution $\boldsymbol{u}^{(P)}$, which satisfies the same equation as $\boldsymbol{u}$, can be written as $u_j^{(P)} = \dfrac{x_j}{2\mu} p$.

Let us verify that this is indeed correct.

$$\nabla \boldsymbol{u}^{(P)} = \frac{\partial}{\partial x_i} u_j^{(P)} = \frac{\partial}{\partial x_i} \left(\frac{x_j}{2\mu} p \right) = \frac{1}{2\mu} \left(\frac{\partial x_j}{\partial x_i} p + x_j \frac{\partial p}{\partial x_i} \right) = \frac{1}{2\mu} \left(\delta_{ij} p + x_j \frac{\partial p}{\partial x_i} \right)$$

Now, take the divergence.

$$\nabla \bullet \left(\nabla \boldsymbol{u}^{(P)} \right) = \frac{1}{2\mu} \left[\frac{\partial}{\partial x_i} \left(\delta_{ij} p \right) + \frac{\partial}{\partial x_i} \left(x_j \frac{\partial p}{\partial x_i} \right) \right]$$

$$= \frac{1}{2\mu} \left[\frac{\partial p}{\partial x_i} \delta_{ij} + \frac{\partial x_j}{\partial x_i} \frac{\partial p}{\partial x_i} + x_j \frac{\partial^2 p}{\partial x_i \partial x_i} \right] = \frac{1}{2\mu} \left[\frac{\partial p}{\partial x_i} \delta_{ij} + \delta_{ji} \frac{\partial p}{\partial x_i} \right] = \frac{1}{\mu} \frac{\partial p}{\partial x_i} = \frac{1}{\mu} \nabla p$$

In obtaining the above result, we have used the fact that the pressure field satisfies Laplace's equation $\left(\dfrac{\partial^2 p}{\partial x_i \partial x_i} = 0 \right)$. We see that the particular solution is indeed correct. Note that we still must satisfy $\nabla \bullet \boldsymbol{u} = 0$. Therefore,

$$\nabla \bullet \boldsymbol{u}^{(H)} = - \nabla \bullet \boldsymbol{u}^{(P)} = - \nabla \bullet \left(\frac{\boldsymbol{x}}{2\mu} p \right) = - \frac{1}{2\mu} \left[3p + \boldsymbol{x} \bullet \nabla p \right]$$

because $\nabla \bullet \left(\frac{\boldsymbol{x}}{2\mu} p \right) = \frac{\partial}{\partial x_i}(x_i p) = \frac{\partial x_i}{\partial x_i} p + x_i \frac{\partial p}{\partial x_i} = \delta_{ii} p + x_i \frac{\partial p}{\partial x_i} = 3p + \boldsymbol{x} \bullet \nabla p$.

Constructing solutions using the invariant representation

Now, we are ready to construct solutions of Stokes problems using invariant representations of the pressure and velocity fields. To do this, we use the following ideas.

1. The pressure field is an absolute scalar, and the velocity field is an absolute vector.

2. The pressure field and the homogeneous part of the velocity field are harmonic functions. If the flow is exterior to a body and the fluid is quiescent far from a body, we can use the decaying harmonics. If the flow is interior to an object, such as a drop, the solution must be finite at the origin which is usually chosen to lie within the object. In this case, we can use growing harmonics.

3. If the object is moving with a velocity U_i or rotating with an angular velocity Ω_i, because of the linearity of the Stokes equation, the fields must also be linear in these vectors.

4. The pressure field and the velocity field can only depend on the position vector and the scalars and vectors arising from the boundary conditions and the shape of the object.

We'll find that there are only a limited number of ways in which the vectors (or scalars) from the boundary conditions can be combined with the set of decaying or growing harmonics using operations such as the cross or dot product or simple scalar multiplication to generate scalars or vectors of order 1. It is this fact that makes it relatively easy to construct the desired solutions in invariant form.

Chapter 8 of Leal (1) contains a detailed discussion of solutions obtained by using the technique outlined here with several examples.

Reference

1. L.G. Leal, Advanced Transport Phenomena: Fluid Mechanics and Convective Transport Processes, Cambridge University Press, 2007.