Notes on the Solution of Stokes's Equation for Axisymmetric Flow in Spherical Polar Coordinates

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The details of the solution of Stokes's Equation for the streamfunction $E^4\psi = 0$ in spherical polar coordinates (r,θ,ϕ) by the method of separation of variables are given in the book "Low Reynolds Number Hydrodynamics" by J. Happel and H. Brenner. The solution is obtained as the sum $\psi^{(1)}(r,\theta)+\psi^{(2)}(r,\theta)$ where $\psi^{(1)}$ satisfies

$$E^2\psi^{(1)} = 0 \tag{1}$$

and $\psi^{(2)}$ is the particular solution of

$$E^2\psi^{(2)} = \psi^{(1)} \tag{2}$$

The solution, in the form of infinite series, is specialized to typical problems in the spherical geometry in a domain that includes at least a portion of the symmetry axis by requiring that the velocity field be bounded along that axis, which corresponds to $\theta = 0, \pi$. The result is given below.

$$\psi(r,\theta) = \sum_{n=2}^{\infty}\left(A_n r^n + B_n r^{-n+1} + C_n r^{n+2} + D_n r^{-n+3}\right)\mathcal{C}_n^{-1/2}(\eta) \tag{3}$$

Here, $\eta = \cos\theta$, and $\mathcal{C}_n^{-1/2}(\eta)$ are the Gegenbauer Polynomials of order n and degree $-1/2$. Their properties are discussed in Sampson (1891), and in Abramowitz and Stegun (1965). The Gegenbauer Polynomials are solutions of the second order ordinary differential equation

$$\left(1-\eta^2\right)\frac{d^2\Phi}{d\eta^2} + n(n-1)\Phi = 0 \tag{4}$$

where n is an integer. The second linearly independent solution of this equation is a function that becomes unbounded along the symmetry axis. The Gegenbauer Polynomials are closely related to the Legendre Polynomials $P_n(\eta)$, which satisfy Legendre's equation.

$$\frac{d}{d\eta}\left[\left(1-\eta^2\right)\frac{d\Phi}{d\eta}\right]+n\left(n+1\right)\Phi=0 \tag{5}$$

Many important properties of the Legendre Polynomials can be found in MacRobert (1967) and in Abramowitz and Stegun (1965).

Some useful relationships involving the Gegenbauer Polynomials are given below.

$$\mathcal{C}_n^{-1/2}\left(\eta\right)=\frac{P_{n-2}\left(\eta\right)-P_n\left(\eta\right)}{2n-1} \tag{6}$$

$$\mathcal{C}_n^{-1/2}\left(\eta\right)=-\int_{-1}^{\eta}P_{n-1}\left(s\right)ds \tag{7}$$

The first few Gegenbauer and Legendre Polynomials are

$\mathcal{C}_0^{-1/2}\left(\eta\right)=1$ $\qquad$ $\mathcal{C}_1^{-1/2}\left(\eta\right)=\eta$

$\mathcal{C}_2^{-1/2}\left(\eta\right)=\frac{1}{2}\left(1-\eta^2\right)$ $\qquad$ $\mathcal{C}_3^{-1/2}\left(\eta\right)=\frac{1}{2}\eta\left(1-\eta^2\right)$

$\mathcal{C}_4^{-1/2}\left(\eta\right)=\frac{1}{8}\left(1-\eta^2\right)\left(5\eta^2-1\right)$

$P_0\left(\eta\right)=1$ $\qquad$ $P_1\left(\eta\right)=\eta$

$P_2\left(\eta\right)=\frac{1}{2}\left(3\eta^2-1\right)$ $\qquad$ $P_3\left(\eta\right)=\frac{1}{2}\eta\left(5\eta^2-3\right)$

$P_4\left(\eta\right)=\frac{1}{8}\left(35\eta^4-30\eta^2+3\right)$

The Gegenbauer Polynomials satisfy the orthogonality property

$$\int_{-1}^{+1}\frac{\mathcal{C}_m^{-1/2}\left(\eta\right)\mathcal{C}_n^{-1/2}\left(\eta\right)}{1-\eta^2}\,d\eta=\frac{2}{n\left(n-1\right)\left(2n-1\right)}\,\delta_{mn} \tag{8}$$

for values of $m,n\geq 2$.

The function $Q_n(\eta)$ used by Leal in the textbook is related to the Gegenbauer Polynomials through $Q_n(\eta) = -C_{n+1}^{-1/2}(\eta)$.

References

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