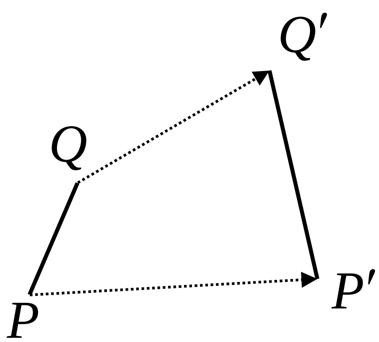
Kinematics of Fluid Motion

R. Shankar Subramanian
Department of Chemical and Biomolecular Engineering
Clarkson University, Potsdam, New York 13699

Kinematics is the study of motion without dealing with the forces that affect motion. The discussion here is of limited scope and for more details, the reader is encouraged to consult any of the references listed at the end. The notation used and the details of the development in many places are directly borrowed from Aris (1) and Batchelor (2). Our focus here is on fluid motion. We shall use rectangular Cartesian coordinates (x, y, z), along with the associated basis set of mutually orthogonal unit vectors $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$. The position vector is labeled $\boldsymbol{x}$.

Imagine a tiny line element $d\boldsymbol{x}$, labeled PQ in the sketch, at some instant of time. After a small amount of time dt, the two ends have moved to new locations because of fluid motion, and the new line element is labeled $P'Q'$.



We can see that if the velocity were to be the same at both ends of the element, it would change neither its length, nor its orientation. Therefore, in a uniform velocity field, there is simple translation of fluid elements with no deformation or rotation. To cause either, the velocity $\boldsymbol{v}(\boldsymbol{x})$ must be non-uniform. To understand the nature of the changes in fluid elements brought about by the flow, we must, therefore, investigate the velocity gradient, $\nabla \boldsymbol{v}$, which is a second order tensor.

From calculus, we know that the differential change dv_x can be written as

$$dv_x = \frac{\partial v_x}{\partial x} dx + \frac{\partial v_x}{\partial y} dy + \frac{\partial v_x}{\partial z} dz$$

and similar results can be written for the changes dv_y and dv_z. It follows that the differential change in the vector velocity, $d\boldsymbol{v}$, is given by

$$d\boldsymbol{v} = \boldsymbol{i}\, dv_x + \boldsymbol{j}\, dv_x + \boldsymbol{k}\, dv_z$$

$$= \boldsymbol{i}\left(\frac{\partial v_x}{\partial x}dx + \frac{\partial v_x}{\partial y}dy + \frac{\partial v_x}{\partial z}dz\right) + \boldsymbol{j}\left(\frac{\partial v_y}{\partial x}dx + \frac{\partial v_y}{\partial y}dy + \frac{\partial v_y}{\partial z}dz\right) + \boldsymbol{k}\left(\frac{\partial v_z}{\partial x}dx + \frac{\partial v_z}{\partial y}dy + \frac{\partial v_z}{\partial z}dz\right)$$

$$= \left(\boldsymbol{i}\frac{\partial v_x}{\partial x} + \boldsymbol{j}\frac{\partial v_y}{\partial x} + \boldsymbol{k}\frac{\partial v_z}{\partial x}\right)dx + \left(\boldsymbol{i}\frac{\partial v_x}{\partial y} + \boldsymbol{j}\frac{\partial v_y}{\partial y} + \boldsymbol{k}\frac{\partial v_z}{\partial y}\right)dy + \left(\boldsymbol{i}\frac{\partial v_x}{\partial z} + \boldsymbol{j}\frac{\partial v_y}{\partial z} + \boldsymbol{k}\frac{\partial v_z}{\partial z}\right)dz$$

$$= \frac{\partial \boldsymbol{v}}{\partial x}dx + \frac{\partial \boldsymbol{v}}{\partial y}dy + \frac{\partial \boldsymbol{v}}{\partial z}dz = \nabla\boldsymbol{v} \bullet d\boldsymbol{x}$$

Thus, the relative velocity of a point a distance $d\boldsymbol{x}$ from any given location is given by the dot product of the tensor $\nabla\boldsymbol{v}$ and the differential line element $d\boldsymbol{x}$. This tensor can be written as follows.

$$\nabla\boldsymbol{v} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{pmatrix}$$

Any tensor can be written as the sum of a symmetric and an antisymmetric tensor. Let us do this with the velocity gradient tensor, writing it as

$$\nabla\boldsymbol{v} = \boldsymbol{E} + \boldsymbol{\Omega}$$

where the (symmetric) rate of strain or rate of deformation tensor $\boldsymbol{E}$ is given by

$$\boldsymbol{E} = \frac{1}{2}\left(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^{\boldsymbol{T}}\right)$$

and the (antisymmetric) vorticity tensor $\boldsymbol{\Omega}$ is given by

$$\boldsymbol{\Omega} = \frac{1}{2}\left(\nabla\boldsymbol{v} - \nabla\boldsymbol{v}^{\boldsymbol{T}}\right)$$

The action of each of these contributions to the velocity gradient will be explored in detail next. First, we consider the vorticity tensor.

Vorticity Tensor $\boldsymbol{\Omega}$

The vorticity tensor $\boldsymbol{\Omega}$ is a skew-symmetric tensor. We can write its components in terms of the components of the velocity gradient as follows.

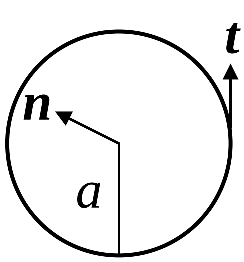
$$\mathbf{\Omega} = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) & 0 \end{pmatrix}$$

A skew-symmetric tensor A_{ij} can be formed from a vector a_k by writing $A_{ij} = \varepsilon_{ijk} a_k$. The vector associated with the vorticity tensor in this manner is $-\frac{1}{2}\boldsymbol{\omega}$, where $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ is known as the vorticity vector. Using the relationship between $\mathbf{\Omega}$ and $-\frac{1}{2}\boldsymbol{\omega}$, we obtain

$\Omega_{ij} dx_j = -\frac{1}{2}\varepsilon_{ijk} dx_j \omega_k$ or in Gibbs notation, $\mathbf{\Omega} \bullet \boldsymbol{dx} = -\frac{1}{2}\boldsymbol{dx} \times \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{dx}$

This means that the relative motion that is contributed by the vorticity tensor at a point an infinitesimal distance away from a reference point in a fluid is that caused by a rigid rotation with an angular velocity equal to $\frac{1}{2}\boldsymbol{\omega}$.

Because a fluid does not usually rotate as a rigid body in the manner that a solid does, we should interpret the above statement as implying that the average angular velocity of a fluid element located at a point is one-half the vorticity vector at that point (2). To prove this claim, consider a surface formed by an infinitesimal circle of radius a located at a point $\boldsymbol{x}$. Let the unit normal vector to the surface (perpendicular to the plane of the paper) be $\boldsymbol{n}$, and the unit tangent vector to the circle at any point be $\boldsymbol{t}$.



Apply Stokes's theorem to the velocity field in this circle.

$$\int_S (\nabla \times \boldsymbol{v}) \bullet \boldsymbol{n}\, dS = \oint_C \boldsymbol{v} \bullet \boldsymbol{t}\, ds$$

Here, dS is an area element on the surface of the circle S and ds is a line element along the circle C. Because $\boldsymbol{v} \bullet \boldsymbol{t}$ is the component of the velocity along the periphery of the circle, we can write the average linear velocity along the circle as $\frac{1}{2\pi a}\oint_C \boldsymbol{v} \bullet \boldsymbol{t}\, ds$ and therefore the average angular velocity as $\frac{1}{2\pi a^2}\oint_C \boldsymbol{v} \bullet \boldsymbol{t}\, ds$. From Stokes's theorem, we see that this is equal to the average value of $\left(\frac{1}{2}\nabla \times \boldsymbol{v}\right) \bullet \boldsymbol{n}$ over the surface of the circle. Thus, in the limit as the radius of the circle approaches zero, we find that the average angular velocity around the circle approaches the value of one-half the component of the vorticity vector in a direction perpendicular to the surface of the circle. We also can show (see Batchelor, page 82) that the angular momentum of a spherical element of fluid is equal to one-half the vorticity times the moment of inertia of the fluid, just as it is for a rigid body.

Vorticity Vector

The vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$, is an important entity in fluid mechanics. It is transported from one place to another in a fluid by convective and molecular means, just as energy and species are, and an appropriate partial differential equation that governs its transport can be written. In addition, vorticity also is intensified by the stretching of vortex lines, a mechanism that is not present in the transport of energy and species. One reason for working with the equations of vorticity transport is that pressure is absent as a dependent variable in those equations. It can be shown that if a fluid mass begins with zero vorticity, and the fluid is inviscid (meaning the viscosity is zero), the vorticity will remain zero in that fluid mass. A flow in which the vorticity is zero is known as an irrotational flow.

Vorticity is generated at fluid-solid interfaces and at fluid-fluid interfaces. Vorticity cannot be generated internally within an incompressible fluid. This is the reason why, in a high Reynolds number flow (implying weak viscous effects) past a rigid body, most of the flow can be described by using the equations that apply to irrotational flow, with the vorticity being confined to a boundary layer near the surface of the body.

Vortex Lines and Tubes

Just as a streamline is a curve to which the velocity vector is tangent everywhere, we can define a vortex line as a curve to which the vorticity is tangent everywhere. If the components of the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ are $(\omega_x, \omega_y, \omega_z)$, then we can write the equations of the space curves that are vortex lines as

$$\frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}.$$

The surface that is formed by all the vortex lines passing through a closed reducible curve is known as a vortex tube. If we construct an open surface *S* bounded by this closed curve

C, we can define the strength of the vortex tube as $\int_S \boldsymbol{dS} \bullet \boldsymbol{\omega}$. By using Stokes's theorem, we can see that this is the circulation $\oint_C \boldsymbol{v} \bullet \boldsymbol{t}\, ds$ where C is any closed curve around the vortex tube, $\boldsymbol{t}$ is a unit tangent vector to the curve at any point, and ds is a line element.

Rate of Strain or Rate of Deformation Tensor $\boldsymbol{E}$

From the above discussion of the vorticity tensor, you can see that the role of that tensor is to describe the instantaneous angular velocity of a fluid element, but that it contributes nothing to deformation of elements. Now, we move on to discuss the significance of the rate of strain tensor, which contains all the information about the deformation.

The rate of strain tensor is a symmetric tensor. We can write its components in terms of the components of the velocity gradient as follows.

$$\mathbf{E} = \begin{pmatrix} \dfrac{\partial v_x}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial v_x}{\partial y} + \dfrac{\partial v_y}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial v_x}{\partial z} + \dfrac{\partial v_z}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v_y}{\partial x} + \dfrac{\partial v_x}{\partial y}\right) & \dfrac{\partial v_y}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v_y}{\partial z} + \dfrac{\partial v_z}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v_z}{\partial x} + \dfrac{\partial v_x}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial v_z}{\partial y} + \dfrac{\partial v_y}{\partial z}\right) & \dfrac{\partial v_z}{\partial z} \end{pmatrix}$$

The diagonal elements of E

Following Aris closely (1), consider a line element $d\boldsymbol{x}$ with a length ds.

$$\frac{d}{dt}\left(ds^2\right) = \frac{d}{dt}\left(d\boldsymbol{x} \bullet d\boldsymbol{x}\right)$$

Using the fact that $\dfrac{d\boldsymbol{x}}{dt} = d\boldsymbol{v}$, the above result can be rewritten as

$$2\, ds \frac{d}{dt}\left(ds\right) = 2\, d\boldsymbol{x} \bullet d\boldsymbol{v} = 2\, d\boldsymbol{x} \bullet \left(\nabla \boldsymbol{v} \bullet d\boldsymbol{x}\right) = 2\, d\boldsymbol{x} \bullet \boldsymbol{E} \bullet d\boldsymbol{x} + 2\, d\boldsymbol{x} \bullet \boldsymbol{\Omega} \bullet d\boldsymbol{x}$$

The second term in the far-right-side is zero because $\boldsymbol{\Omega}$ is an antisymmetric tensor. To see this, we write

$$d\boldsymbol{x} \bullet \boldsymbol{\Omega} \bullet d\boldsymbol{x} = \Omega_{ij} dx_i dx_j \;=\; \Omega_{ji} dx_j dx_i \;=\; -\Omega_{ij} dx_i dx_j \text{ so that } \Omega_{ij} dx_i dx_j = 0.$$

In the above result, after writing the result in index notation, we first exchange the indices i and j to obtain an intermediate result, and then use the antisymmetry property to write $\Omega_{ij} = -\Omega_{ji}$.

Therefore, we find that

$ds \frac{d}{dt}(ds) = d\boldsymbol{x} \bullet \boldsymbol{E} \bullet d\boldsymbol{x}$ from which, by dividing through by ds^2 we can write

$$\frac{1}{ds}\frac{d}{dt}(ds) = \frac{d\boldsymbol{x}}{ds} \bullet \boldsymbol{E} \bullet \frac{d\boldsymbol{x}}{ds}$$

The vector $d\boldsymbol{x}/ds$ is a unit vector pointing in the direction of the infinitesimal vector $d\boldsymbol{x}$. Therefore, we can think of the right side of the above result as the "double projection" of the tensor $\boldsymbol{E}$ in that direction. The term "projection" is used in a loose sense here. The physical meaning is clear. The rate of strain of a line element pointing in any direction at a given point (which is the time rate of change of length, divided by the length) is the dot product of a unit vector in that direction with the dot product of the rate of strain tensor with the same unit vector. Let us choose the direction to be the $x-$direction. In this case, the rate of strain of a line element in that direction is simply E_{11}, which is equal to $\frac{\partial v_x}{\partial x}$. In a like manner, the rate of strain of a line element in the $y-$direction is $\frac{\partial v_y}{\partial y}$, and that in the $z-$direction is $\frac{\partial v_z}{\partial z}$. This is the physical interpretation of the diagonal elements of the rate of strain tensor.

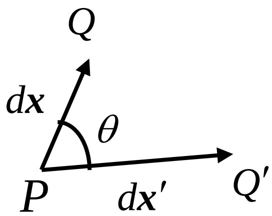
The sum of the diagonal elements of $\boldsymbol{E}$, known as the trace of $\boldsymbol{E}$ is $\nabla \bullet \boldsymbol{v}$. This is known as the rate of dilatation of a fluid element at the given location. To see why, consider a material body occupying a volume V enclosed by the surface S. Let us inquire how V changes with time. We can write the rate of change of the volume of a material body with time as the integral of $d\boldsymbol{S} \bullet \boldsymbol{v}$ over the surface.

$\frac{dV}{dt} = \int_S d\boldsymbol{S} \bullet \boldsymbol{v} = \int_V \nabla \bullet \boldsymbol{v}\, dV$ by the divergence theorem.

From the above, we can see that $\lim_{V \to 0} \frac{1}{V}\frac{dV}{dt} = \lim_{V \to 0} \frac{1}{V}\int_V \nabla \bullet \boldsymbol{v}\, dV = \nabla \bullet \boldsymbol{v}$. So, the trace of $\boldsymbol{E}$ is the rate of increase in the volume of an infinitesimal element, divided by its volume, and is called the rate of dilatation. When the flow is incompressible, the rate of dilatation is zero.

The off-diagonal elements of $\boldsymbol{E}$

Now, consider two line elements $d\boldsymbol{x}$ and $d\boldsymbol{x}'$ at a given point $\boldsymbol{x}$ and let the angle between them be θ.



Let us investigate the time rate of change of the dot product of the vectors $d\boldsymbol{x}$ and $d\boldsymbol{x}'$.

$$\begin{aligned}\frac{d}{dt}\left(ds\, ds'\cos\theta\right) &= \frac{d}{dt}\left(d\boldsymbol{x}\bullet d\boldsymbol{x}'\right) = dv_i dx_i' + dx_i dv_i' \\ &= \frac{\partial v_i}{\partial x_j} dx_j dx_i' + \frac{\partial v_i}{\partial x_j} dx_j' dx_i\end{aligned}$$

In writing the result in the second term in the second line, we have used the fact that the infinitesimal change dv_i' is the change in the velocity over an infinitesimal distance in the direction of the vector $d\boldsymbol{x}'$. Interchanging the indices i and j in that second term permits us to combine the two terms.

$$\frac{d}{dt}\left(ds\, ds'\cos\theta\right) = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) dx_i' dx_j = 2\,E_{ij}\; dx_i' dx_j$$

Dividing both sides by $ds\, ds'$ yields

$$\frac{1}{ds\, ds'}\frac{d}{dt}\left(ds\, ds'\cos\theta\right) \;=\; 2\,E_{ij}\,\frac{dx_i'}{ds'}\frac{dx_j}{ds} = 2\,\frac{d\boldsymbol{x}'}{ds'}\bullet\boldsymbol{E}\bullet\frac{d\boldsymbol{x}}{ds}$$

So, we see that if we take the dot product of $\boldsymbol{E}$ with unit vectors in two different directions in succession (the order is immaterial because $\boldsymbol{E}$ is symmetric), the result is the left side of the above equation. Let us work out the differentiation in the left side.

$$\frac{1}{ds\, ds'}\frac{d}{dt}\left(ds\, ds'\cos\theta\right) = \cos\theta\left[\frac{1}{ds}\frac{d}{dt}\left(ds\right) + \frac{1}{ds'}\frac{d}{dt}\left(ds'\right)\right] - \sin\theta\,\frac{d\theta}{dt}$$

The term in square brackets in the right side is the sum of the individual rates of strain of the two line elements. We can see that the above result reduces to the earlier result we obtained when the two vectors dx_i and dx_i' are the same. Let us consider the case when the two vectors are orthogonal to each other. In this case, we obtain

$$\frac{d\boldsymbol{x}'}{ds'} \bullet \boldsymbol{E} \bullet \frac{d\boldsymbol{x}}{ds} = -\frac{1}{2}\frac{d\theta}{dt}$$

So, the sequential dot products of $\boldsymbol{E}$ with unit vectors in two orthogonal directions yields one-half the rate of decrease of the angle between the unit vectors in those directions. If we choose these two orthogonal directions to coincide with any two coordinate directions, then the dot products yield the off-diagonal elements of $\boldsymbol{E}$. For example, if we use x and $y-$ directions, the element is $E_{12}\left(=E_{21}\right)$. Similar physical interpretations can be given to the other off-diagonal elements of the rate of strain tensor. Thus, the off-diagonal elements describe shear deformation of the fluid.

There are three mutually orthogonal directions associated with the symmetric tensor $\boldsymbol{E}$ that are known as its eigenvector or principal directions. We can use a basis set built from these principal directions to describe the components of the tensor. If we do, the tensor will be diagonal. The off-diagonal elements will be zero, so that the rate of change of the angles between the principal directions is zero; of course the entire set of principal axes can rotate, and in fact it does, with the angular velocity $\frac{1}{2}\boldsymbol{\omega} = \frac{1}{2}\nabla \times \boldsymbol{v}$.

Instantaneous Deformation of a Fluid Element

Based on all of the above material on kinematics, we can conclude that in a flow, an infinitesimal spherical element of fluid undergoes translation, rotation, and deformation in general. It deforms into an ellipsoid whose axes are aligned with the principal axes of the rate of strain tensor. This ellipsoid also rotates with an instantaneous angular velocity that is equal to the one-half of the vorticity of the fluid at the given point.

Some good sources for further study are listed below.

References

1. R. Aris, Vectors, Tensors, and the Basic Equations of Fluid Mechanics, Dover Publications, 1989, Chapter 4 (original by Prentice-Hall, 1962).

2. G.K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, 1967, Chapter 5.

3. C. Truesdell, Kinematics of Vorticity, Indiana University Press, 1954.

4. J. Serrin, Mathematical Principles of Classical Fluid Dynamics, Handbuch der Physik VIII/1, Ed. S. Flugge, 1959.

5. P.G. Saffman, Vortex Dynamics, Cambridge University Press, 1992.