The Gamma Function

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The Gamma function $\Gamma(x)$ is sometimes encountered in the analysis of transport problems. Its definition is given below.

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{1}$$

Using this definition, we can write

$$\Gamma(x+1) = \int_0^\infty t^{x} e^{-t} dt \tag{2}$$

If we integrate the right side of Equation (2) by parts, we obtain the following important result.

$$\begin{aligned} \Gamma(x+1) &= \int_0^\infty t^{x} e^{-t} dt = \left[-t^x e^{-t} \right]_0^\infty + x \int_0^\infty t^{x-1} e^{-t} dt \\ &= x\Gamma(x) \end{aligned} \tag{3}$$

as long as $x > 0$. Let us now examine the definition of the factorial of an integer.

$$(n)! = n \times (n-1) \cdots 2 \times 1 = n \times (n-1)! \tag{4}$$

Therefore, if we consider the idea that $\Gamma(x+1) = x!$ when x takes on integer values, we see that the Gamma function is a generalization of the factorial. If we have a table of values of $\Gamma(x)$ for $0 < x \le 1$, we can obtain the value of $\Gamma(x)$ for any $x \ge 1$ using Equation (3). Some specific values are $\Gamma(1/3) \approx 2.67894$; $\Gamma(1/2) = \sqrt{\pi}$; $\Gamma(2/3) \approx 1.35412$.

Reference

M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, Dover, 1965, Chapter 6.