Bessel functions

R. Shankar Subramanian
Department of Chemical and Biomolecular Engineering
Clarkson University, Potsdam, New York 13699

Bessel functions are solutions of the differential equation given below.

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(\alpha^2 - p^2\right)y = 0$$

obtained using Frobenius series. The two linearly independent solutions are $J_p(\alpha x)$ and $J_{-p}(\alpha x)$ so long as p is not an integer. These are known as the Bessel functions of the first kind. If p is zero, the two solutions are identical, and if p is a positive integer, the second solution $J_{-p}(\alpha x)$ is a linear multiple of the first solution $J_p(\alpha x)$. In such cases, the second linearly independent solution of Bessel's equation is written as $Y_p(\alpha x)$, which is known as the Bessel function of the second kind. In the following, the factor α is omitted wherever it is not needed. The series solution for $J_p(x)$ can be written as

$$J_p(x) = \sum_{k=0}^{\infty}\frac{(-1)^k (x/2)^{2k+p}}{k!(k+p)!}$$

The quantity $(k+p)!$ is to be interpreted as the gamma function $\Gamma(k+p+1)$ when p is not an integer.

When p is 0 or a positive integer, we can write $Y_p(x)$ as shown below.

$$Y_0(x) = \frac{2}{\pi}\left[\left(\log\frac{x}{2}+\gamma\right)J_0(x) + \sum_{k=0}^{\infty}(-1)^{k+1}\varphi(k)\frac{(x/2)^{2k}}{(k!)^2}\right]$$

and for a positive integer n,

$$Y_n(x) = \frac{2}{\pi}\left[\begin{array}{l}\left(\log\frac{x}{2}+\gamma\right)J_n(x) - \frac{1}{2}\sum_{k=0}^{n-1}\frac{(n-k-1)!(x/2)^{2k-n}}{k!} \\ \qquad + \frac{1}{2}\sum_{k=0}^{\infty}(-1)^{k+1}\left[\varphi(k)+\varphi(k+n)\right]\frac{(x/2)^{2k+n}}{k!(n+k)!}\end{array}\right]$$

where $\varphi(k)=\sum_{m=1}^{k}\frac{1}{m}$ and $\varphi(0)=0$. The symbol γ stands for Euler's constant, defined as $\gamma=\lim_{k\to\infty}\left[\varphi(k)-\log k\right]=0.5772157...$

If p is not zero or a positive integer, the function $Y_p(x)$ can be written as

$$Y_p(x)=\frac{(\cos p\pi)J_p(x)-J_{-p}(x)}{\sin p\pi}$$

which is, of course, a linear combination of the two linearly independent solutions. This definition reduces to those given above when p approaches an integer value.

If $Z_n(\alpha x)=c_1J_n(\alpha x)+c_2Y_n(\alpha x)$ where c_1 and c_2 are arbitrary constants, the following relationships are applicable.

$$\frac{d}{dx}Z_n(\alpha x)=\alpha Z_{n-1}(\alpha x)-\frac{n}{x}Z_n(\alpha x)$$

$$\frac{d}{dx}Z_n(\alpha x)=-\alpha Z_{n+1}(\alpha x)+\frac{n}{x}Z_n(\alpha x)$$

$$\frac{d}{dx}\left[x^nZ_n(\alpha x)\right]=\alpha x^nZ_{n-1}(\alpha x)$$

$$\frac{d}{dx}\left[x^{-n}Z_n(\alpha x)\right]=-\alpha x^{-n}Z_{n+1}(\alpha x)$$

Also, you will find the following variations useful.

$$Z_{n-1}(\alpha x)+Z_{n+1}(\alpha x)=\frac{2n}{\alpha x}Z_n(\alpha x)$$
$$Z_{n-1}(\alpha x)-Z_{n+1}(\alpha x)=\frac{2}{\alpha}\frac{d}{dx}\left[Z_n(\alpha x)\right]$$

In the interval (a,b) where $0<a<b$, the orthogonality property of Bessel functions can be stated as follows:

$$\int_a^b xZ_p(\lambda_m x)Z_p(\lambda_n x)\,dx=0,\qquad m\neq n$$

$$\int_a^b xZ_p^2(\lambda_n x)\,dx = \left[\frac{x^2}{2}\left\{\left(1-\frac{p^2}{\lambda_n^2 x^2}\right)Z_p^2(\lambda_n x) + Z_p'^2(\lambda_n x)\right\}\right]_a^b$$

provided λ_n is a real root of

$$h_1\lambda Z_{p+1}(\lambda b) - h_2 Z_p(\lambda b) = 0$$

and two numbers k_1 and k_2 exist (both not zero) such that for all values of n,

$$k_1\lambda_n Z_{p+1}(\lambda_n a) - k_2 Z_p(\lambda_n a) = 0$$

The prime that appears in the right side of the result for the integral is to be interpreted as differentiating with respect to the entire argument, not just with respect to x. For example, whereas $\frac{d}{dx} J_0(\lambda x) = -\lambda\, J_1(\lambda x)$, the result for $J_0'(\lambda x) = -J_1(\lambda x)$.

The above results for the integrals hold in the case when $a = 0$, provided we set $c_2 = 0$. This means that when $a = 0$, the results apply only for the Bessel function of the first kind, and not the second kind. This is because $Y_n(x)$ becomes singular as $x \to 0$.

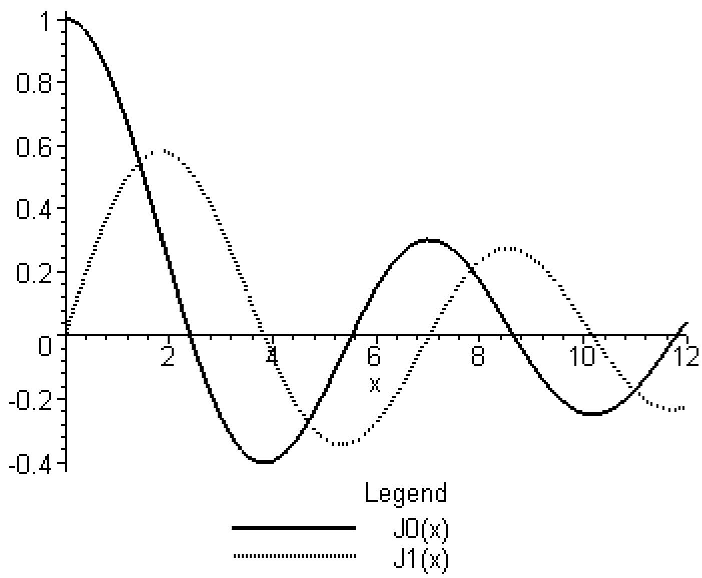
Given below are some specialized versions of the general result for $\int_a^b xZ_p^2(\lambda_n x)\,dx$ when $Z_p(\lambda_n x) = J_p(\lambda_n x)$, from Hildebrand (1).

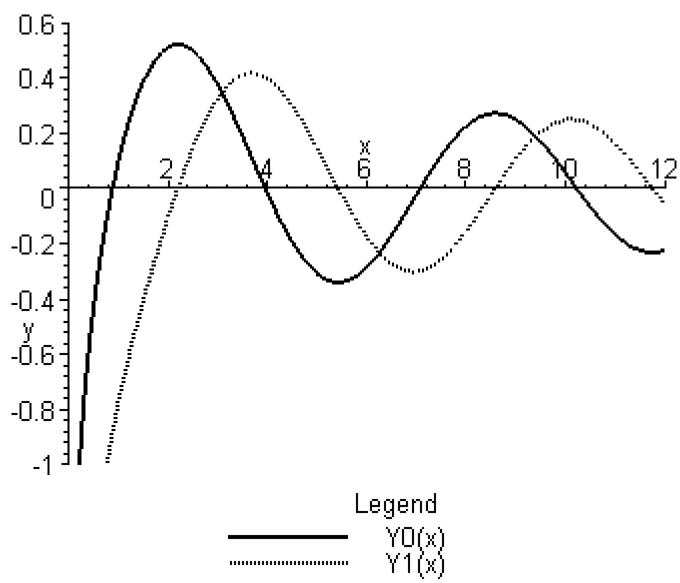
If $J_p(\lambda_n) = 0$, then $\int_0^1 xJ_p^2(\lambda_n x)\,dx = \frac{1}{2} J_{p+1}^2(\lambda_n)$

If $J_p'(\lambda_n) = 0$, then $\int_0^1 xJ_p^2(\lambda_n x)\,dx = \frac{1}{2}\frac{\lambda_n^2 - p^2}{\lambda_n^2} J_p^2(\lambda_n)$

If $kJ_p(\lambda_n) + \lambda_n J_p'(\lambda_n) = 0$, then $\int_0^1 xJ_p^2(\lambda_n x)\,dx = \frac{1}{2}\frac{\lambda_n^2 - p^2 + k^2}{\lambda_n^2} J_p^2(\lambda_n)$

Here are sample graphs showing the behavior of the functions $J_0(x)$ and $J_1(x)$, followed by $Y_0(x)$ and $Y_1(x)$.





References

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