The Alternating Unit Tensor ε_{ijk}

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The alternating unit tensor ε_{ijk} is useful when expressing certain results in compact form in index notation. It is defined as follows.

$\varepsilon_{ijk} = 0$ if any two of the indices are equal

$\varepsilon_{ijk} = +1$ when the indices form an even permutation of (123)

$\varepsilon_{ijk} = -1$ when the indices form an odd permutation of (123)

When the indices are different from each other, they must form either an even or an odd permutation of (123). We determine whether a permutation is even or odd by examining the number of transpositions that will lead to it. For example, consider (231). One way to obtain this is to first swap 1 and 2 in (123) to yield (213) and then swap 1 and 3 to get (231). We have gone through two such swaps. Therefore, this is an even permutation of (123). On the other hand, to get (213) from (123), we needed just a single swap. Hence, (213) is an odd permutation of (123).

Using the above information, you can establish that among the 27 components of this third order tensor, 21 are zero. Of the remaining 6, three take on the value of +1, and the other three, the value –1.

By enumeration, prove to yourself that

$$\varepsilon_{ijk}\,\varepsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

Here are some examples illustrating the uses of the alternating unit tensor.

Cross Product of Two Vectors

$$\boldsymbol{a} \times \boldsymbol{b} = \varepsilon_{ijk} a_i b_j$$

Determinant of a second-order tensor $\boldsymbol{A}$

$$\text{Det}\left(A_{ij}\right) = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}$$