Lagrange and Stokes Streamfunctions

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When a two-dimensional flow is incompressible, so that the equation of continuity reduces to $\nabla \bullet \boldsymbol{v} = 0$, where $\boldsymbol{v}$ represents the vector velocity field, it is sometimes advantageous to define a scalar field termed the streamfunction. It is typically represented by the symbol ψ. Commonly, one solves the governing equation for the single variable ψ, along with the associated boundary conditions, instead of solving the Navier-Stokes and continuity equations for the velocity and pressure fields. Once the streamfunction field ψ is determined, it is possible to infer the velocity components and the pressure distribution. The price to be paid for this simplification is that the streamfunction satisfies a fourth-order partial differential equation, in contrast with the Navier-Stokes equation, which is a second order partial differential equation for the velocity field.

In this brief document, we shall learn about the definition and the physical significance of the streamfunction. For further details, the reader should consult a textbook, such as that by Happel and Brenner (1973).

Lagrange Streamfunction

When the flow is two-dimensional on a plane (chosen as the $x-y$ plane in a rectangular cartesian coordinate system), the Lagrange streamfunction ψ is related to the velocity field as follows.

$$\boldsymbol{v} = \nabla \psi \times \boldsymbol{i}_z = \boldsymbol{i}_x \frac{\partial \psi}{\partial y} - \boldsymbol{i}_y \frac{\partial \psi}{\partial x}$$

where $\boldsymbol{i}_x$ and $\boldsymbol{i}_y$ and $\boldsymbol{i}_z$ are unit vectors in the x, y, and $z-$directions, respectively.

When we say "two-dimensional" flow, we mean that the velocity component in the third direction is identically zero, and furthermore that the two remaining velocity components are independent of the coordinate in that third direction. In this case, the third direction is the $z-$direction, so that $v_z \equiv 0$, and $\frac{\partial \boldsymbol{v}}{\partial z} = \mathbf{0}$.

Recall that the cross-product of two vectors is perpendicular to the plane containing those two vectors. Thus, from the definition equation for the streamfunction, we infer that the velocity vector is perpendicular to the $z-$direction, and therefore lies on the $x-y$ plane. In addition, we also see that it must be perpendicular to the vector $\nabla \psi$. Because $\nabla \psi$ is normal to level curves of the streamfunction ψ, this implies that the velocity vector must be tangent everywhere to level curves of the streamfunction, which are also known as streamlines. There can be no flow normal to a streamline.

We can use the definition of ψ to infer the relationship between the velocity components and the streamfunction. We can write

$$\boldsymbol{v} = \boldsymbol{i}_x v_x + \boldsymbol{i}_y v_y = \boldsymbol{i}_x \frac{\partial \psi}{\partial y} - \boldsymbol{i}_y \frac{\partial \psi}{\partial x}$$

so that

$$\boxed{v_x = \frac{\partial \psi}{\partial y}} \qquad \boxed{v_y = -\frac{\partial \psi}{\partial x}}$$

Consider incompressible two-dimensional flow over a flat plate as shown below.

At a given location x along the plate, if we integrate the result $\frac{\partial \psi}{\partial y} = v_x$ between the solid surface and a given y, we obtain

$$\psi(x, y) = \psi(x, 0) + \int_0^y v_x(x, y)\, dy$$

Because of the kinematic condition at the solid surface, ψ is constant along the solid surface, and it is customary to assign it a value of zero at that surface. Thus,

$$\boxed{\psi(x, y) = \int_0^y v_x(x, y)\, dy}$$

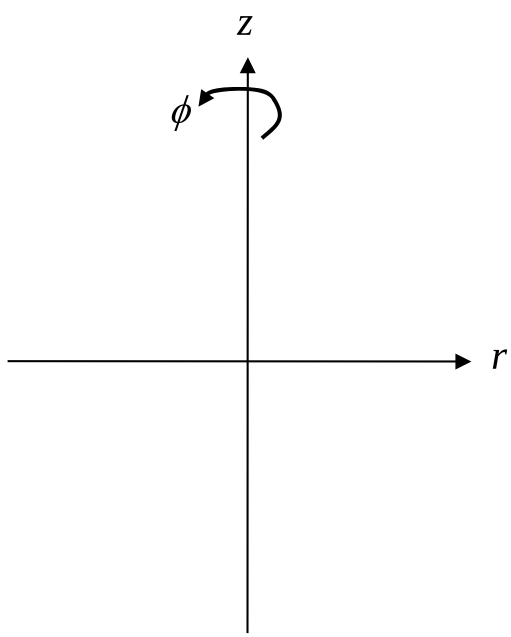
implying that the streamfunction represents the volumetric flow rate per unit width in the $x-$direction in this situation. Note that the constant value assigned to $\psi(x, 0)$ is immaterial, because only gradients of ψ matter.

Stokes Streamfunction

When a sphere settles in a fluid, the flow it creates is not strictly two-dimensional, but is axially symmetric. Thus, we define a different streamfunction to handle situation like this.

When a flow is symmetric about the $z-$axis, we can still regard the flow as “two-dimensional” in the sense that the velocity component in the azimuthal direction ϕ measured around the axis is identically zero, and furthermore $\dfrac{\partial \boldsymbol{v}}{\partial \phi} = \boldsymbol{0}$. Thus, in analogy with the flow on a plane discussed earlier, we can define a streamfunction for axisymmetric flows. Such a streamfunction also is represented by the symbol ψ, but is distinguished from the Lagrange streamfunction by designating it the “Stokes streamfunction.”

The sketch below shows the $z-$axis, along with the azimuthal coordinate ϕ. Also shown is a cylindrical polar coordinate r, which is the distance measured from the axis.



The Stokes streamfunction ψ is related to the velocity vector field $\boldsymbol{v}$ as follows.

$$\boldsymbol{v} = \frac{1}{r}\left(\boldsymbol{i}_\phi \times \nabla \psi\right)$$

where $\boldsymbol{i}_\phi$ is a unit vector in the $\phi-$direction. Again, recognizing that the cross-product of $\boldsymbol{i}_\phi$ and $\nabla \psi$ must be perpendicular to the plane containing them, it is straightforward to infer that the velocity field must lie on planes of constant ϕ, known as meridian planes. This is logical, because the velocity component v_ϕ is identically zero, and the velocity field is independent of the azimuthal coordinate ϕ. Also, the vector velocity must be tangent to level curves of the Stokes streamfunction ψ on the meridian plane, which are therefore, known as streamlines.

If we select any point on a meridian plane and join it by any curve to the axis, then rotate this curve through a complete revolution around the axis (by an angle 2π), we create a bowl-like surface. The volumetric flow rate Q through this surface is related to the streamfunction at the selected point via $Q = 2\pi\psi$. The sign convention chosen here is that Q is measured downward in the sketch of the coordinate system displayed earlier.

In cylindrical polar coordinates, the velocity components are related to the streamfunction as follows.

$$\boxed{v_r = \frac{1}{r}\frac{\partial \psi}{\partial z}} \qquad \boxed{v_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}}$$

Likewise, the relationship between the Stokes streamfunction and the velocity components v_r and v_θ in a spherical polar coordinate system (r,θ,ϕ) can be inferred to be

$$\boxed{v_r = -\frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}} \qquad \boxed{v_\theta = \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}}$$

To learn more details about the streamfunction, including the partial differential equation and typical boundary conditions it satisfies, you can consult Happel and Brenner (1973).

Reference

J. Happel and H. Brenner, **Low Reynolds Number Hydrodynamics**, Noordhoff International, Leyden, 1973; originally published by Prentice-Hall in 1965.