Elements of Prandtl's Boundary Layer Theory

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The failure of potential flow (incompressible irrotational flow) theory to predict drag on objects when a fluid flows past them provided the impetus for Prandtl to put forward a theory of the boundary layer adjacent to a rigid surface. Prandtl's principal assumptions are listed below.

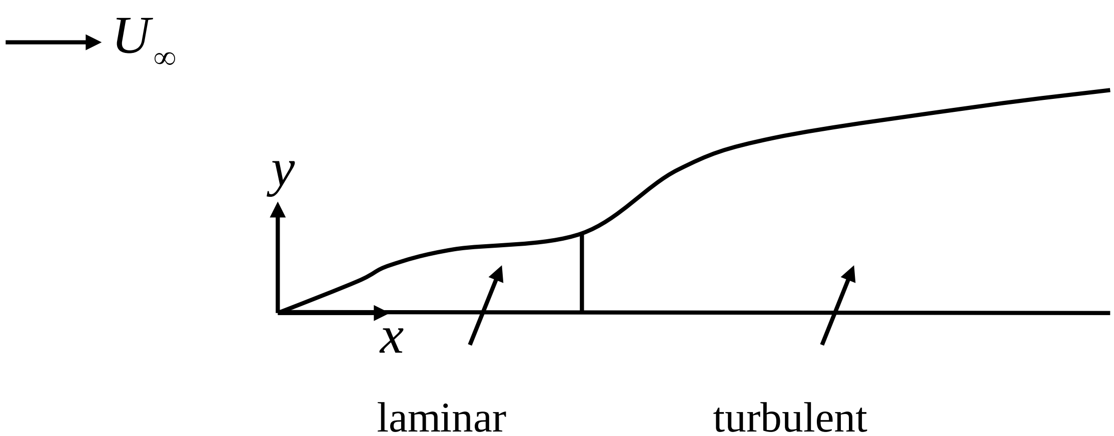
Assumptions

1. When a fluid flows past an object at large values of the Reynolds number, the flow region can be divided into two parts.

(i) Away from the surface of the object, viscous effects can be considered negligible, and potential flow can be assumed.

(ii) In a thin region near the surface of the object, called the boundary layer, viscous effects cannot be neglected, and are as important as inertia.

2. The pressure variation can be calculated from the potential flow solution along the surface of the object, neglecting viscous effects altogether, and assumed to be impressed upon the boundary layer.



Transition from laminar to turbulent flow in the boundary layer on a flat plate occurs at $\mathrm{Re}_x \approx 5\times10^5$, where $\mathrm{Re}_x = (xU_\infty)/\nu$. Here, ν is the kinematic viscosity of the fluid.

The assumptions can be used to establish the order of magnitude of the boundary layer thickness.

A typical inertia term in the Navier-Stokes equation in rectangular Cartesian coordinates is $\rho u \frac{\partial u}{\partial x}$, and a typical viscous term is $\mu \frac{\partial^2 u}{\partial y^2}$. Here, (u,v) are the velocity components in the (x,y) directions, and ρ and μ are the density and the dynamic viscosity of the fluid. We can estimate the order of magnitude of each of these terms for a plate of length L as follows.

$$\rho u \frac{\partial u}{\partial x} \sim \rho \frac{U_\infty^2}{L} \qquad\qquad \mu \frac{\partial^2 u}{\partial y^2} \sim \mu \frac{U_\infty}{\delta^2}$$

Because the viscous force in the boundary layer is of comparable order to the inertia force, these two order estimates must be comparable.

$\rho \frac{U_\infty^2}{L} \sim \mu \frac{U_\infty}{\delta^2}$ or $\delta^2 \sim \frac{\mu L}{\rho U_\infty}$, which can be recast as

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{\mathrm{Re}_L}}$$

where the Reynolds number based on the length of the plate $\mathrm{Re}_L = \frac{LU_\infty}{\nu}$.

This type of argument is called a **scaling analysis**. It is a valuable tool in dealing with transport problems. You can see that it provides not only an idea of the variables on which key quantities depend, but also the form of this dependence without having to solve the partial differential equations involved.

In a like manner, we can find a scale estimate of the drag as well. The shear stress at the plate surface is $\tau_w = \mu \frac{\partial u}{\partial y}(x,0)$. We can estimate the order of this quantity as $\tau_w(x) = \mu \frac{U_\infty}{\delta}$. Because the shear stress is a local quantity, we should use an order of magnitude of the variation of the boundary layer thickness δ with x. From the order of magnitude argument used earlier, we can estimate it as $\delta(x) \sim \frac{x}{\sqrt{\mathrm{Re}_x}} = \sqrt{\frac{\nu x}{U_\infty}}$. If the width of the plate in the $z-$direction is w, the drag on the plate surface is given by

$$D = w\int_0^L \tau_w(x)\,dx = \mu w \frac{U_\infty^{3/2}}{\nu^{1/2}} \int_0^L \frac{dx}{\sqrt{x}}.$$

Ignoring the numerical factor of 2 that appears after performing the integration (because we are only estimating the order of magnitude), we can write

$$D \sim w\sqrt{\rho\,\mu U_\infty^3\,L}$$

A rigorous calculation from boundary layer theory yields the result

$$D = 0.664\, w\sqrt{\rho\, \mu U_\infty^3\, L}$$

confirming the correctness of our scaling argument.

The Displacement Thickness

The displacement thickness of the boundary layer is defined as the distance by which the potential flow streamlines are displaced by the presence of the boundary layer. We can construct a mathematical definition in the case of the flat plate by recognizing that the displacement thickness δ_1 is that thickness of the uniform stream that accounts for the "lost" flow because of the presence of the solid surface.

$$U_\infty \delta_1 = \int_0^\infty \left(U - u\right) dy$$

or

$$\delta_1 = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy$$

Order of magnitude analysis of the continuity and Navier-Stokes Equations

Now, we shall go through an order of magnitude analysis of the two-dimensional Navier-Stokes equations for steady incompressible Newtonian laminar flow over a flat plate and simplify them using Prandtl's ideas. For more details, you can consult Schlichting [1].

We shall use scaled variables, using L as a reference length, and U_∞ as a reference velocity. The symbols x and y are used for the scaled counterparts of the physical coordinates in the sketch, and the symbols u and v are used for the dimensionless counterparts of the physical velocity components in the x and y directions, respectively. The scaled incompressible version of the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

From the scaling, we know that u is $O(1)$. This means that the magnitude of u lies between 0 and a number that is of the order of unity. In this particular case, because the maximum value of the physical velocity is that of the uniform stream approaching the plate, namely U_∞, the maximum value of u is, in fact, precisely 1. But this is not necessarily the meaning implied by the order symbol that we are using.   Note that the order of magnitude of a quantity is the same regardless of its sign.

Because the velocity u varies in the range mentioned above, while the scaled variable x also varies from 0 to 1 (we say $x \sim O(1)$), we can conclude that the derivative $\frac{\partial u}{\partial x}$ is $O(1)$ as well. From the continuity equation, we see that $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ must sum to zero; this forces the derivative $\frac{\partial v}{\partial y}$ to be $O(1)$. We know that the variable $y \sim O(\delta)$ where δ represents the boundary layer thickness divided by the length L. In other words, δ is the scaled boundary layer thickness. Because the derivative $\frac{\partial v}{\partial y} \sim O(1)$, we must conclude that the change in the scaled velocity component v across the boundary layer must be of $O(\delta)$. We know from the kinematic condition that $v = 0$ at the surface $y = 0$. Therefore, the magnitude of v must of $O(\delta)$. We note that δ is a very small quantity when the Reynolds number $\mathrm{Re}_L \gg 1$. We express this fact by stating $\delta \ll 1$. Therefore, the scaled velocity in the $y-$direction in the boundary is a very small quantity.

Now, following Schlichting [1] we proceed to use similar arguments in the two components of the Navier-Stokes equation applicable to this situation. First, consider the $x-$component.

$$
\begin{array}{ccccccccc}
u\,\dfrac{\partial u}{\partial x} & + & v\,\dfrac{\partial u}{\partial y} & = & -\dfrac{\partial p}{\partial x} & + & \dfrac{1}{\mathrm{Re}_L} & \left[\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right] & \\
1 \quad 1 & & \delta \quad \dfrac{1}{\delta} & & & & \delta^2 & 1 \qquad \dfrac{1}{\delta^2} &
\end{array}
$$

Below each term in the equation, we have written the order of magnitude of that term. We already have discussed the order of magnitude of u, v, and $\frac{\partial u}{\partial x}$. To estimate the order of magnitude of $\frac{\partial u}{\partial y}$, we first note that u varies from 0 to 1 across the boundary layer, while the variable y varies from 0 to δ. This is the reason for the estimate that $\frac{\partial u}{\partial y} \sim O\left(\frac{1}{\delta}\right)$. To estimate the order of magnitude of the second derivatives, we must use similar arguments. For example, consider the derivative $\frac{\partial^2 u}{\partial x^2}$. We know that $\frac{\partial u}{\partial x} \sim O(1)$. So, this quantity must change from 0 to a magnitude of the order of unity in a scaled distance x that also changes from 0 to 1. This is the reason for estimating the order of $\frac{\partial^2 u}{\partial x^2}$ as being unity. In a like manner, the derivative $\frac{\partial u}{\partial y} \sim O\left(\frac{1}{\delta}\right)$, which means that it varies from 0 to $1/\delta$ across the boundary layer, in a distance of the order δ. Therefore, the second derivative $\frac{\partial^2 u}{\partial y^2} \sim O\left(\frac{1}{\delta^2}\right)$. The order of magnitude of the Reynolds number was established earlier on page 2.

Comparing the two viscous terms, we see that the viscous force in the $x-$direction is negligible when compared to that in the $y-$direction. We need to retain all the other terms in the $x-$component momentum equation because they are all of comparable order of magnitude.

Now, let us consider the $y-$component of the Navier-Stokes equation.

$$u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}_L}\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

$$1 \quad \delta \qquad \delta \quad 1 \qquad\qquad\qquad \delta^2 \qquad \delta \qquad \frac{1}{\delta}$$

The order of magnitude of the derivatives has been estimated in the same manner as outlined earlier. Once again, we see that the viscous transport of $y-$momentum in the $x-$direction is much weaker than that in the $y-$direction, and can be neglected. The most important aspect of the above equation is that all the retained terms are of $O(\delta)$, so that the pressure gradient $\frac{\partial p}{\partial y}$ must necessarily be of the same order (or smaller). Because the variation of pressure in the $y-$direction in the boundary layer must occur over a distance of $O(\delta)$, it is evident that the scaled pressure change across the thickness of the boundary layer $\Delta p \sim O(\delta^2)$. This is very small, and can be ignored, which is Prandtl's assumption 2 listed on page 1. Because the pressure change across the boundary layer is negligible, the pressure distribution along the surface of the object, evaluated from the potential flow, can be assumed to be "impressed" on the boundary layer. This means that $\frac{\partial p}{\partial x}$ in the $x-$component momentum equation is a known inhomogeneity, and we can simply ignore the $y-$component momentum equation because all the terms are small.

Summarizing the above, we have found from the scaling analysis that the viscous term in the main direction of flow (x) is negligible compared with the viscous term in the direction normal to the solid surface. Furthermore, the pressure gradient in the $x-$component momentum equation is established from potential flow theory and evaluated along the surface of the object, and the $y-$component momentum equation is neglected. Thus, we have two equations for the two unknown velocity components.

Even though our analysis assumed a flat plate, you can see that for a thin boundary layer, the effects of curvature of the surface would be negligible at leading order. Therefore, as long as we define x and y as distance coordinates along and normal to a surface, respectively, the same equations can be written for flow past an object with a curved surface. For convenience, Prandtl's steady two-dimensional boundary layer equations for incompressible Newtonian flow are written in physical variables below. To avoid clutter, we have retained the same symbols for the velocities and coordinates as those used earlier for scaled variables, but this should not be a source of confusion.

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Navier-Stokes Equation

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\,\frac{\partial^2 u}{\partial y^2}$$

For the flat plate problem, the potential flow is simply $u = U_\infty$. This means that the potential flow pressure gradient is zero. Therefore, the Navier-Stokes equation simplifies to

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = \nu\,\frac{\partial^2 u}{\partial y^2}$$

The boundary conditions are written as follows.

$$\begin{aligned} &u(0,y) = U_\infty \\ &u(x,0) = 0 \\ &v(x,0) = 0 \\ &u(x, y\to\infty) \to U_\infty \end{aligned}$$

Commonly, the last condition is replaced with $u(x,\infty) = U_\infty$.

Note that

1. The important nonlinear (inertial) terms have been retained.

2. The number of differential equations has been reduced from three to two, consistent with the simplification that the pressure distribution is “known” from potential flow theory.

3. Because the variation of pressure across the boundary layer is negligible to this order of approximation, the potential flow pressure distribution can be evaluated right at the solid surface and used as a known inhomogeneity in the boundary layer equations.

Reference

1. H. Schlichting, Boundary Layer Theory, McGraw-Hill, 1968.