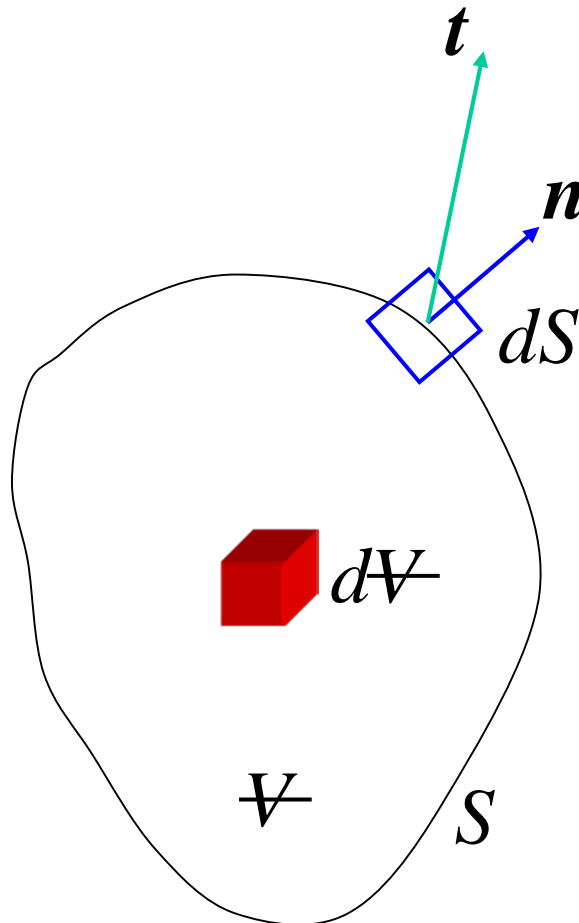


# Navier-Stokes Equation: Principle of Conservation of Momentum

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Newton formulated the principle of conservation of momentum for rigid bodies. It took some time for the corresponding version for a continuum, representing a fluid, to be developed. The result is attributed to Cauchy, and is known as Cauchy's equation (1). A derivation of Cauchy's equation is given first. Then, by using a Newtonian constitutive equation to relate stress to rate of strain, the Navier-Stokes equation is derived.

The principle of conservation of momentum is applied to a fixed volume of arbitrary shape in space that contains fluid. This volume is called a "Control Volume." Fluid is permitted to enter or leave the control volume. A control volume  $\mathcal{V}$  is shown in the sketch.



Also marked on the sketch is the bounding surface  $S$  of this control volume, called the control surface; an element of surface area  $dS$  and the unit outward normal (vector) to that area element,

$\mathbf{n}$  are shown as well. In addition, the vector stress  $\mathbf{t}$  exerted by the neighboring fluid on the fluid in the control volume on the area element  $dS$  also is displayed. The stress vector will be discussed in more detail shortly. We note that the vector symbol  $d\mathbf{S} = \mathbf{n} dS$  is used to represent a directed differential area element on the surface.

We begin with a verbal statement of the principle of conservation of momentum.

**Rate of increase of momentum of material within the control volume = Net rate at which momentum enters the control volume with the flowing fluid + Sum of the forces acting on the fluid in the control volume**

Next, we shall develop a mathematical representation of the above statement. Because the density  $\rho$  of the fluid and the velocity  $\mathbf{v}$  can both depend on position, to determine the rate of increase of momentum in the control volume, first it is necessary to determine the mass content in the differential volume element  $dV$ . Multiplying the differential volume by the density at that location gives the amount of mass in the differential volume element as  $\rho dV$ , and multiplying this mass by the velocity at that location gives the momentum content of the fluid in the differential volume element. The total momentum in the control volume  $V$  is obtained by integrating this product over the entire control volume. Therefore,

$$\text{Momentum content of the fluid in the control volume} = \int_V \rho \mathbf{v} dV$$

The time rate of change of this momentum content of the fluid in the control volume is obtained by differentiating the above result with respect to time. Because the control volume is fixed in space, the time derivative can be taken inside the integral and becomes a partial derivative in time, obtained while keeping spatial coordinates fixed. Thus, the left side in the verbal statement of the

principle of conservation of momentum is  $\int_V \frac{\partial}{\partial t} (\rho \mathbf{v}) dV$ , where  $t$  represents time.

Next, we need to develop a result for the net rate of entry of momentum with the fluid flowing into and out of the control volume through the control surface. For this, we need to consider the differential area element  $dS$ . The rate of influx of mass into the control volume through this differential area is given by  $-\rho(\mathbf{v} \cdot \mathbf{n}) dS$ . Therefore, the rate of influx of momentum with the fluid flowing into the control volume through the area element is  $-\rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n}) dS$ . Integrating this result over the entire control surface leads to the following result.

Net rate of influx of momentum into the control volume with the flow =

$$-\int_S \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n}) dS = -\int_S (\rho \mathbf{v} \mathbf{v}) \cdot d\mathbf{S}$$

Using the divergence theorem, the right side of the above equation becomes  $-\int_V \nabla \cdot (\rho \mathbf{v} \mathbf{v}) dV$ .

Now, we turn our attention to summing the forces acting on the fluid in the control volume. These can be broadly divided into body forces and contact forces. Body forces act on every volume

element within the fluid occupying the control volume at any given instant, and do not need contact with the material to exert their influence. The most common example of a body force is the gravitational force on objects. Other examples include electrical or magnetic forces. Let us use the symbol  $\mathbf{f}$  to designate the body force per unit mass acting on the fluid in the control volume. In the case of the gravitational force exerted by Earth on objects,  $\mathbf{f} = \mathbf{g}$ , where  $\mathbf{g}$  is the acceleration due to gravity vector, pointed toward the center of mass of Earth. It is straightforward to work out the result for the body force on the fluid in the control volume. Once again, we begin with the differential volume element  $dV$ . The mass of fluid in this volume element is  $\rho dV$  so that the body force on this mass is  $\rho \mathbf{f} dV$ . Adding up all the contributions from such volume elements in the control volume we obtain the following result.

$$\text{Body force acting on the fluid in the control volume} = \int_{\mathcal{V}} \rho \mathbf{f} dV$$

The contact force on the fluid located at the surface of the control volume arises from intermolecular forces exerted by the molecules on the outside of the surface on the molecules on the inside of the surface. This force, expressed as a result per unit area, is termed the stress  $\mathbf{t}$ . The magnitude and direction of this vector will, in general, depend on the location, as well as the orientation of the area element, given by the direction of the unit normal  $\mathbf{n}$ . Multiplying the stress  $\mathbf{t}$  by the area  $dS$  yields a contact force on the area element that is equal to  $\mathbf{t} dS$ . Adding up all the contributions over the surface of the control volume leads to the following result for the total contact force acting on the fluid in the control volume.

$$\text{Contact force acting on the fluid in the control volume} = \int_S \mathbf{t} dS$$

Therefore, the principle of conservation of momentum can be cast in mathematical form as follows:

$$\int_{\mathcal{V}} \frac{\partial}{\partial t} (\rho \mathbf{v}) dV = - \int_{\mathcal{V}} \nabla \cdot (\rho \mathbf{v} \mathbf{v}) dV + \int_{\mathcal{V}} \rho \mathbf{f} dV + \int_S \mathbf{t} dS$$

Now, consider shrinking  $\mathcal{V}$  to zero, and achieving this by permitting a characteristic length scale  $\ell$  to approach zero. The volume integrals will approach zero proportional to  $\ell^3$ , whereas the surface area will approach zero proportional to  $\ell^2$ . Therefore, if we divide through by the surface area, we can write

$$\lim_{\ell \rightarrow 0} \frac{1}{S} \left[ \int_{\mathcal{V}} \left( \frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) - \rho \mathbf{f} \right) dV \right] = 0$$

Hence, we can conclude that

$$\lim_{\ell \rightarrow 0} \left[ \frac{1}{S} \int_S \mathbf{t} dS \right] = 0$$

This implies that the stresses on a fluid are in local equilibrium. This result can be used to establish that the stress vector on an area element at any location in a fluid is given by the equation  $\mathbf{t} = \mathbf{n} \bullet \mathbf{T}$ , where  $\mathbf{T}$  is a second order tensor, called the stress tensor in the fluid at that point. For details regarding how this can be established, you can consult pages 99-101 of Aris (1). Furthermore, by invoking the principle of conservation of angular momentum, it can be shown that except in rare cases that need not concern us, the stress tensor is symmetric. Substituting this new result for the stress vector allows us to rewrite the result for the total contact force acting on the fluid in the control volume as  $\int_S \mathbf{n} \bullet \mathbf{T} dS$ , or  $\int_S dS \bullet \mathbf{T}$ . Upon using the divergence theorem, this can be shown

to be equal to the volume integral  $\int_V \nabla \bullet \mathbf{T} dV$ .

Using the results developed above, we can write the following mathematical statement representing the principle of conservation of momentum applied to a control volume.

$$\int_V \frac{\partial}{\partial t}(\rho \mathbf{v}) dV = - \int_V \nabla \bullet (\rho \mathbf{v} \mathbf{v}) dV + \int_V \rho \mathbf{f} dV + \int_V \nabla \bullet \mathbf{T} dV$$

This result can be rewritten as follows.

$$\int_V \left[ \frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \bullet (\rho \mathbf{v} \mathbf{v}) - \rho \mathbf{f} - \nabla \bullet \mathbf{T} \right] dV = \mathbf{0}$$

We wish to conclude that at every point in the fluid, the integrand of the above result must be zero. To do this we can use the same arguments employed in deriving the equation of conservation of mass. That is, the control volume is arbitrary, and the integrand is a continuous vector function of position. Thus, with each component of the vector function, we can argue that if it is non-zero at any location in the fluid, then the control volume can be selected as the neighborhood of that point in which it retains the same sign, leading to a violation of the above integral balance. Therefore, the integrand must be zero at every point in the fluid.

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \bullet (\rho \mathbf{v} \mathbf{v}) - \rho \mathbf{f} - \nabla \bullet \mathbf{T} = \mathbf{0}$$

or

$$\boxed{\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \bullet (\rho \mathbf{v} \mathbf{v}) = \rho \mathbf{f} + \nabla \bullet \mathbf{T}}$$

This is known as Cauchy's equation. We can combine the two terms in the left side. Using index notation for convenience,

$$\begin{aligned} \frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \bullet (\rho \mathbf{v} \mathbf{v}) &= \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_j v_i) \\ &= v_i \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_i}{\partial t} + v_i \frac{\partial}{\partial x_j}(\rho v_j) + \rho v_j \frac{\partial v_i}{\partial x_j} \end{aligned}$$

$$\begin{aligned}
&= v_i \left[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_j) \right] + \rho \left[ \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right] \\
&= \mathbf{v} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] + \rho \left[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right]
\end{aligned}$$

where the first term in square brackets is zero because of the continuity equation. Thus, we find

$$\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = \rho \left[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right] = \rho \frac{d\mathbf{v}}{dt}$$

where  $\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$  is the material derivative of the velocity, namely, the derivative with respect to time taken while travelling with the fluid at any given point. Therefore, Cauchy's equation can be rewritten as

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} + \nabla \cdot \mathbf{T}$$

When the body force is Earth's gravity, we can write Cauchy's equation as

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{g} + \nabla \cdot \mathbf{T}$$

where  $\mathbf{g}$  was defined earlier as the acceleration due to gravity vector.

In index notation, Cauchy's equation is written as follows.

$$\rho \frac{dv_j}{dt} = \rho f_j + \frac{\partial T_{ij}}{\partial x_i}$$

## Navier-Stokes Equation

The continuity (or conservation of mass) equation and Cauchy's equation are insufficient by themselves, because we have too many unknowns. The density and the components of the velocity vector field constitute four unknowns, while the scalar conservation of mass equation and the vector conservation of momentum equation provide four scalar balances. But, we also have six unknown components of the symmetric stress tensor in the conservation of momentum equation. Thus, in order to pose a solvable system of equations, we need to have additional information. There is no other conservation principle we can use. Therefore, we must resort to a "constitutive equation" known as a phenomenological model, to describe the connection between the stress tensor and the components of the velocity vector. It seems plausible to postulate that the stress in a fluid  $\mathbf{T}$  is related to the velocity gradient  $\nabla \mathbf{v}$ , which can be written as the sum of a symmetric tensor, namely the rate of deformation  $\mathbf{E}$ , and an antisymmetric tensor, namely the vorticity  $\mathbf{\Omega}$ . Most derivations postulate that the stress must only depend upon the rate of deformation  $\mathbf{E}$ ,

because the vorticity in a fluid  $\boldsymbol{\Omega}$  describes only the instantaneous angular motion of an element of fluid, and does not have anything to do with deformation. Others argue that a “principle of material frame indifference” requires that the same constitutive equation apply to a fluid whether one considers a laboratory reference frame or one that is instantaneously rotating at the local angular velocity at a point in the fluid. This argument has been contested in the literature. Aris (1) and Batchelor (2) provide an alternative viewpoint, postulating a general tensorial connection between  $\boldsymbol{T}$  and  $\nabla\boldsymbol{v}$ , and showing that isotropy arguments lead to the exclusion of  $\boldsymbol{\Omega}$  from the relationship, leading to a dependence of the state of stress only on the rate of deformation. In any case, this is where we begin our development, namely, by postulating that  $\boldsymbol{T}$  is some function of  $\boldsymbol{E}$ . As discussed in Aris (1) and Slattery (3), one can take advantage of the fact that both the stress and rate of deformation are symmetric tensors, and write the most general connection between two symmetric tensors in the following form.

$$\boldsymbol{T} = \chi_0 \boldsymbol{I} + \chi_1 \boldsymbol{E} + \chi_2 \boldsymbol{E} : \boldsymbol{E}$$

Here,  $\chi_0, \chi_1$ , and  $\chi_2$  are functions of the three invariants of the tensor  $\boldsymbol{E}$ . If you are wondering about higher order terms in what appears to be an expansion in power series, it is shown in Aris (1) that higher powers of  $\boldsymbol{E}$  can be reduced to results that depend only on the lower powers included here and the three invariants of  $\boldsymbol{E}$ .

The task now is to establish the nature of the functions in the relationship between  $\boldsymbol{T}$  and  $\boldsymbol{E}$ . Here, we simplify the problem greatly by assuming that the elements of the stress tensor depend only linearly on the elements of the rate of deformation tensor. This is known as a Newtonian relationship, to honor Newton who first postulated a linear dependence of the shear stress on the velocity gradient in a one-dimensional context. The first invariant of  $\boldsymbol{E}$  is the trace of  $\boldsymbol{E}$ , which we know is the divergence of the velocity field  $\nabla \cdot \boldsymbol{v}$ . This, of course, is linear in the elements of  $\boldsymbol{E}$ . The second invariant is quadratic in the elements of  $\boldsymbol{E}$ , and the third invariant, which is the determinant of  $\boldsymbol{E}$ , is cubic in the elements of  $\boldsymbol{E}$ . Thus, neither the second nor the third invariant can appear in the relationship between the stress and the rate of deformation for a Newtonian fluid. Furthermore, we can see that the function  $\chi_2$  must be zero because of the quadratic dependence of  $\boldsymbol{E} : \boldsymbol{E}$  on the elements of  $\boldsymbol{E}$ ,  $\chi_1$  has to be a scalar constant independent of the elements of  $\boldsymbol{E}$ , and  $\chi_0$  can, at best, be a linear function of  $\nabla \cdot \boldsymbol{v}$ . Using this information, the Newtonian constitutive equation is written as follows.

$$\boldsymbol{T} = [-p + \lambda \nabla \cdot \boldsymbol{v}] \boldsymbol{I} + 2\mu \boldsymbol{E}$$

Here,  $p$  will be seen to be the pressure field in the fluid shortly, and  $\lambda$  and  $\mu$  are identified as physical properties of a given fluid. The symbol  $\mu$  stands for a physical property known as the coefficient of shear viscosity, or simply viscosity. Sometimes, it is called dynamic viscosity to distinguish it from another property known as kinematic viscosity. There appears to be no specific name for the property represented by the symbol  $\lambda$ , but there is a name for  $\kappa = \lambda + \frac{2}{3}\mu$ . It is called the coefficient of bulk viscosity and is used to account for dissipation that occurs in a fluid that is rapidly expanded and contracted. In incompressible flow, the velocity field is solenoidal so

that this type of dissipation does not occur, making the value of the coefficient of bulk viscosity moot. Historically, the so-called Stokes hypothesis assumed that  $\kappa = 0$ , but this is not necessarily true. In any case, the value of  $\kappa$  will prove to be unimportant for incompressible flow, which implies  $\nabla \cdot \mathbf{v} = 0$ .

We can see that in the absence of motion  $\mathbf{v} \equiv \mathbf{0}$ , so that  $\mathbf{E} \equiv \mathbf{0}$  as well. Thus, the result for the stress reduces to

$$\mathbf{T} = -p\mathbf{I}$$

That is, the state of stress in a stationary fluid is isotropic. The stress vector on any area element at a given location is given by

$$\mathbf{t} = \mathbf{n} \cdot \mathbf{T} = -p(\mathbf{n} \cdot \mathbf{I}) = -pn$$

At a given point, regardless of the orientation of the area element, the stress is of the same magnitude  $p$ , termed pressure. Pressure is usually positive, so that the stress in a stationary fluid points in the direction opposite to the normal vector to the area element. Thus, a positive pressure results in compressive stress on an element of fluid at any given location in the fluid.

To describe the more general case of a moving fluid, we first need to insert the Newtonian constitutive equation into the Cauchy equation. Thus, we write

$$\nabla \cdot \mathbf{T} = \nabla \cdot \left\{ [-p + \lambda \nabla \cdot \mathbf{v}] \mathbf{I} + 2\mu \mathbf{E} \right\} = -\nabla p + \nabla(\lambda \nabla \cdot \mathbf{v}) + 2\nabla \cdot (\mu \mathbf{E})$$

The viscosity  $\mu$  is commonly assumed to be a constant to simplify the governing equation. We find  $\nabla \cdot \mathbf{E}$  as follows.

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\partial E_{ij}}{\partial x_i} = \frac{1}{2} \left[ \frac{\partial}{\partial x_i} \left( \frac{\partial v_i}{\partial x_j} \right) + \frac{\partial}{\partial x_i} \left( \frac{\partial v_j}{\partial x_i} \right) \right] \\ &= \frac{1}{2} \left[ \frac{\partial}{\partial x_j} \left( \frac{\partial v_i}{\partial x_i} \right) + \frac{\partial}{\partial x_i} \left( \frac{\partial v_j}{\partial x_i} \right) \right] = \frac{1}{2} [\nabla(\nabla \cdot \mathbf{v}) + \nabla \cdot (\nabla \mathbf{v})] \end{aligned}$$

We shall only be concerned with incompressible flow so that we can set  $\nabla \cdot \mathbf{v} = 0$ . Substituting for  $\nabla \cdot \mathbf{E}$  in the result for  $\nabla \cdot \mathbf{T}$  and using the assumptions stated here yields

$$\nabla \cdot \mathbf{T} = -\nabla p + \mu \nabla \cdot (\nabla \mathbf{v}) = -\nabla p + \mu \nabla^2 \mathbf{v}$$

Now, we can return to the Cauchy equation and substitute the above result for  $\nabla \cdot \mathbf{T}$  in the right side to obtain the following version of the principle of conservation of momentum that we shall use extensively.

$$\boxed{\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{v}} \quad \text{or alternatively}$$

$$\underbrace{\rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right)}_{\text{inertia force}} = \underbrace{-\nabla p}_{\text{pressure force}} + \underbrace{\rho \mathbf{g}}_{\text{gravitational force}} + \underbrace{\mu \nabla^2 \mathbf{v}}_{\text{viscous force}}$$

This equation was first derived by Navier using molecular arguments, and later by Stokes for a continuum. Thus, it is known as the Navier-Stokes equation for incompressible flow and constant viscosity. It is a vector balance in which each term has the dimensions of force per unit volume. The left side is the inertia force. The first term in the right side represents the pressure force, the second the gravitational force, and the third, the viscous force. The viscosity is a physical property, and therefore depends on the thermodynamic state of the fluid. It is sensitive to temperature in both gases and liquids, and relatively insensitive to pressure under commonly encountered conditions, with the exception of unusually large pressures. For more information, you can consult Bird et al. (4). Also you will find a good introduction to non-Newtonian constitutive relations in Chapter 8 of Bird et al.

## References

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