Continuity Equation

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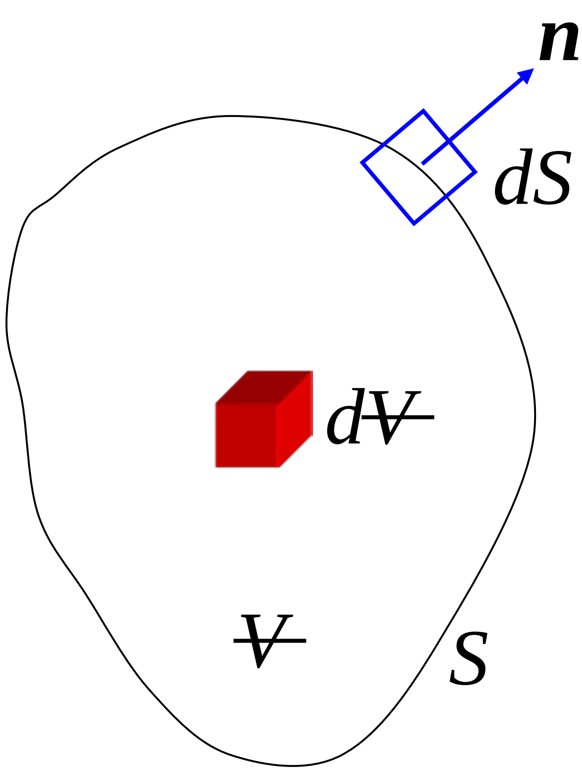
Based on observation, one can postulate the idea that mass is neither created nor destroyed. In other words, it is conserved. This is termed the Principle of Conservation of Mass. This principle is applied to a fixed volume of arbitrary shape in space that contains fluid. This volume is called a "Control Volume." Fluid is permitted to enter or leave the control volume.

A control volume V is shown in the sketch.



Also marked on the sketch is the bounding surface S of this control volume, called the control surface; an element of surface area dS and the unit outward normal (vector) to that area element, $\boldsymbol{n}$ are shown as well. The vector symbol $\boldsymbol{dS} = \boldsymbol{n}\, dS$ is used to represent a directed differential area element on the surface.

Just like the principle of conservation of mass, one can make similar statements about energy and momentum, being careful to accommodate ways in which energy or momentum can enter or leave a fixed volume in space occupied by a fluid. These conservation statements are put in mathematical form and termed "integral balances." These balances are useful in a variety of problems. Here, we shall apply the principle of conservation of mass to the control volume shown in the sketch, and eventually obtain a partial differential equation commonly known as the continuity equation. We begin with a verbal statement of the principle of conservation of mass.

Rate of increase of mass of material within the control volume = Net rate at which mass enters the control volume.

Let us write a mathematical representation of the above statement. To determine the rate of accumulation of mass in the control volume, we begin with the mass content in the differential volume element $d\mathcal{V}$, because the density ρ of the fluid can depend on position. Multiplying the differential volume by the density at that location gives the amount of mass in the differential volume element, and the total mass M in the control volume $\mathcal{V}$ is obtained by integrating this product over the entire control volume. Therefore,

$$M = \int_{\mathcal{V}} \rho \, d\mathcal{V}$$

The time rate of change of this mass content in the control volume is dM / dt. Because the control volume is fixed in space, the time derivate can be taken inside the integral and becomes a partial derivate in time, obtained when keeping spatial coordinates fixed. Thus, the left side in the verbal statement of the principle of conservation of mass is $\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\mathcal{V}$, where t represents time.

Now, we need to develop a result for the net rate of entry of mass into the control volume through the control surface. For this, we consider the differential area element dS. Now, let us define the mass flux through space as the vector $\boldsymbol{N}_M$. Then the rate of entry of mass into the control volume through the area element dS is $\left(-\boldsymbol{N}_M \bullet \boldsymbol{n}\, dS\right)$, which is $\left(-\boldsymbol{N}_M \bullet \boldsymbol{dS}\right)$. The total rate of entry of mass over the entire surface can be written as the integral $\int_S \left(-\boldsymbol{N}_M \bullet \boldsymbol{dS}\right)$. This is the right side in the verbal statement of the principle of conservation of mass. Equating the two sides yields the following result.

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\mathcal{V} = -\int_S \boldsymbol{N}_M \bullet \boldsymbol{dS} \quad \text{or} \quad \boxed{\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\mathcal{V} + \int_S \boldsymbol{N}_M \bullet \boldsymbol{dS} = 0}$$

A theorem that applies to vector fields permits us to convert a surface integral such as the one in the above equation into a volume integral. It is known as the Gauss divergence theorem or Green's theorem. The vector field must satisfy conditions regarding continuity of derivatives, and all the fields that we encounter are assumed to satisfy these conditions. Using this theorem, we can write the following result.

$$\int_S \boldsymbol{N}_M \bullet \boldsymbol{dS} = \int_{\mathcal{V}} \left(\nabla \bullet \boldsymbol{N}_M\right) d\mathcal{V}$$

Using this equality, we can rewrite the principle of conservation of mass as

$$\int_{V}\left[\frac{\partial\rho}{\partial t}+\nabla\bullet\boldsymbol{N}_M\right]dV=0$$

We wish to conclude that at every point in the fluid, the integrand of the above result must be zero. To do this we need to use two important ideas. One is that our control volume is arbitrary in shape and location. The second is that the fields ρ, $\boldsymbol{N}_M$, and their derivatives are all continuous functions of position. Thus, the integrand is a continuous function of position. This means that if the integrand is non-zero at any point in space, we are guaranteed a neighborhood of that point in which it retains the same sign. Then, we can consider that specific neighborhood the control volume, in which case the integral will be non-zero, violating the integral balance stated above. This precludes the integrand being non-zero anywhere in the fluid. Thus, it must be zero at every point in the fluid.

$$\frac{\partial\rho}{\partial t}+\nabla\bullet\boldsymbol{N}_M=0$$

The mass flux is the product of the density and the volume flux, which is the velocity.

$$\boldsymbol{N}_M=\rho\boldsymbol{v}$$

Substituting this result in the partial differential equation yields the continuity equation.

$$\frac{\partial\rho}{\partial t}+\nabla\bullet(\rho\boldsymbol{v})=0$$

By working out the divergence of the product, we can rewrite this as

$$\frac{\partial\rho}{\partial t}+\boldsymbol{v}\bullet\nabla\rho+\rho(\nabla\bullet\boldsymbol{v})=0$$

It is common practice to combine the first two terms in the left side and write the result as the material derivative of density with respect to time.

$$\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+\boldsymbol{v}\bullet\nabla\rho$$

Bird et al. (1) use the symbol $\frac{D\rho}{Dt}$ for the material derivative. We shall use $\frac{d\rho}{dt}$ in our work. The material derivative is the time derivative taken while keeping the material coordinates fixed. Physically, this means that it is the time derivative obtained while moving with a material particle, i.e. moving with the flow.

Thus, the continuity equation is rewritten as

$$\frac{d\rho}{dt} + \rho\left(\nabla \bullet \boldsymbol{v}\right) = 0$$

The assumption of incompressible flow, implying that the density of an element of fluid does not change with a change in pressure, is used commonly. With this assumption, the continuity equation reduces to

$$\boxed{\nabla \bullet \boldsymbol{v} = 0}$$

In rectangular Cartesian coordinates (x, y, z) this is written as follows.

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Appropriate representations in other coordinate systems can be found in textbooks such as that by Bird et al. (1).

The Assumption of Incompressible Flow

Incompressible flow implies that the variation in the density of an element of fluid due to changes in pressure can be considered negligible. Because pressure variations are encountered in fluids, one might wonder about the limits on the validity of this assumption. Liquids are virtually incompressible, displaying very small variations in density in response to pressure changes, so that incompressible flow is almost always an excellent assumption in liquids. Gases, on the other hand, undergo a significant density change when the pressure is changed. In the flow of gases, the assumption of incompressible flow can be used so long as

$$\left|\frac{\Delta\rho}{\rho}\right| \ll 1$$

where $\Delta\rho$ represents a typical change in density encountered in the flow, and the symbol $\ll$ stands for "much less than."

As shown in pages 9-11 of Schlichting (2) this condition implies that the square of the ratio of a characteristic velocity in the fluid v_0 to the speed of sound c in the fluid, is much less than unity.

$$\left(\frac{v_0}{c}\right)^2 \ll 1$$

The Mach number M is defined as

$$M = \frac{v_0}{c}$$

so that the requirement is that $M^2 \ll 1$. In practical terms, this means that the assumption of incompressible flow in a gas is good so long as the Mach number is relatively small, that is, so long as the typical flow velocities are much smaller than the speed of sound in the fluid.

A more detailed discussion of the conditions for the validity of the assumption of incompressible flow, accounting also for thermal expansion, can be found in pages 167-171 of Batchelor (3).

References

1. R.B. Bird, W.E. Stewart, and E.N. Lightfoot, Transport Phenomena, Wiley, 2007.

2. H. Schlichting, Boundary Layer Theory, McGraw-Hill, 1968.

3. G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, 1967.