

Unsteady Heat Transfer: Lumped Thermal Capacity Model

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A variety of chemical engineering situations involve unsteady heating or cooling. Consider a batch reactor, for example, in which the reactants are introduced and permitted to react. If the reaction is exothermic, it may be necessary to remove heat, and endothermic reactions require heat to be added. Another example is that of a hot object that is taken out of a furnace and cooled in air. A third example might be the situation that occurs when we measure our body temperature by inserting a thermometer tip under the tongue. The thermometer tip is initially at room temperature, and gradually approaches the temperature prevalent under the tongue.

In the above situations, in addition to depending on position within the mass of fluid or solid under consideration, the temperature also depends on time. Such problems are, by nature, quite complicated, and engineers have devised means of simplifying them when conditions warrant. One such simplification is that introduced by the lumped thermal capacity model. In this model, when a mass, which can be well-conducting solid or a well-mixed fluid, is subjected to heating or cooling by exposure to an environment with which it exchanges heat, one assumes that temperature variations within the mass can be neglected in comparison with the temperature difference between the mass and the surrounding fluid. Another way of stating this idea is that the resistance to heat transfer within the mass is negligible when compared with the resistance to heat transfer with the surroundings.

We begin with an unsteady energy balance on a mass m of well-conducting solid or well-mixed fluid with a constant specific heat c . We assume the volume of this mass to remain constant. As the temperature of this mass changes, its specific heat will change, but if the range of temperature variation is moderate, we can use an average value of the specific heat and treat it as a constant for the purpose of this analysis. The energy content of the mass can be written as $mc(T - T_{\text{reference}})$ where $T_{\text{reference}}$ is a suitable reference temperature for the purpose of this definition. Its value will not matter, as you will see shortly. Now, assume that the mass is surrounded by a fluid at a temperature T_{∞} to which it loses heat. Let the heat transfer coefficient between the mass and the fluid be h , and the surface area of the mass be A .

In words, the unsteady balance reads as follows:

Time rate of change of the energy content of a given mass = Rate at which energy is added to the mass + Rate of production of energy within the mass

In typical situations such as when a solid mass at a high temperature is taken out of a furnace and cooled in a fluid, there is no production of energy within the mass. Thus, we can set that term to zero. Using the symbols we have defined already, we can then write the unsteady energy balance as follows:

$$\frac{d}{dt} [mc(T - T_{\text{reference}})] = -hA(T - T_{\infty})$$

When T is greater than T_{∞} , the mass is losing heat to the surroundings. Thus, the rate of addition is the negative of the rate of heat loss, which explains the minus sign on the right side. This equation is correct whether the mass is hotter than the surroundings or cooler than the surroundings, as you can verify for yourself.

Because the product mc and the reference temperature T_{∞} are constant, we can rewrite the above equation as

$$mc \frac{dT}{dt} = -hA(T - T_{\infty})$$

For the purpose of solving this first order linear ordinary differential equation for the temperature, we first recast it in standard form.

$$\frac{dT}{dt} + \left[\frac{hA}{mc} \right] T = \frac{hA}{mc} T_{\infty}$$

Because this is a first-order differential equation, when we integrate it, there will be an arbitrary constant in the solution. To evaluate it, we'll need an initial condition on the temperature of the mass. That is, we need to know the initial temperature of the mass. Let it be T_0 . We signify the fact that the temperature T is a function of time by writing it as $T(t)$. Then, we can write this initial condition as

$$T(0) = T_0$$

It is straightforward to integrate the differential equation for the temperature as a function of time and apply the initial condition to evaluate the arbitrary constant. This is accomplished using an integrating factor. The integrating factor is $\exp\left[\frac{hA}{mc}t\right]$. For convenience, let us define the constant $\alpha = hA/(mc)$ so that the integrating factor is $e^{\alpha t}$ and the differential equation is

$$\frac{dT}{dt} + \alpha T = \alpha T_{\infty}$$

Multiply both sides of the differential equation by $e^{\alpha t}$.

$$e^{\alpha t} \left(\frac{dT}{dt} + \alpha T \right) = \alpha T_{\infty} e^{\alpha t}$$

The left side can be rewritten as a derivative of the product of the integrating factor and the temperature. Therefore,

$$\frac{d}{dt} (Te^{\alpha t}) = \alpha T_{\infty} e^{\alpha t}$$

We can integrate both sides immediately. The result is

$$Te^{\alpha t} = \alpha T_{\infty} \int e^{\alpha t} dt + C$$

where C is an arbitrary constant of integration. Performing the indefinite integration and rearranging slightly leads to the following result.

$$T = T_{\infty} + C e^{-\alpha t}$$

Apply the initial condition $T(0) = T_0$. This means evaluating both sides of the equation at $t = 0$.

$$T_0 = T_{\infty} + C$$

so that the constant C can be obtained as

$$C = T_0 - T_{\infty}$$

Substitute this result for C in the solution.

$$T = T_{\infty} + (T_0 - T_{\infty}) e^{-\alpha t}$$

Slightly rearranging this result, and substituting for α , we can write the solution as follows.

$$\boxed{\frac{T - T_{\infty}}{T_0 - T_{\infty}} = \exp\left(-\frac{hA}{mc} t\right)}$$

If you check carefully, you'll notice that $\alpha = hA/(mc)$ has dimensions of inverse time. Therefore, $(hA/mc)t$ is dimensionless. Let us define

$$\boxed{\tau = \frac{hA}{mc} t}$$

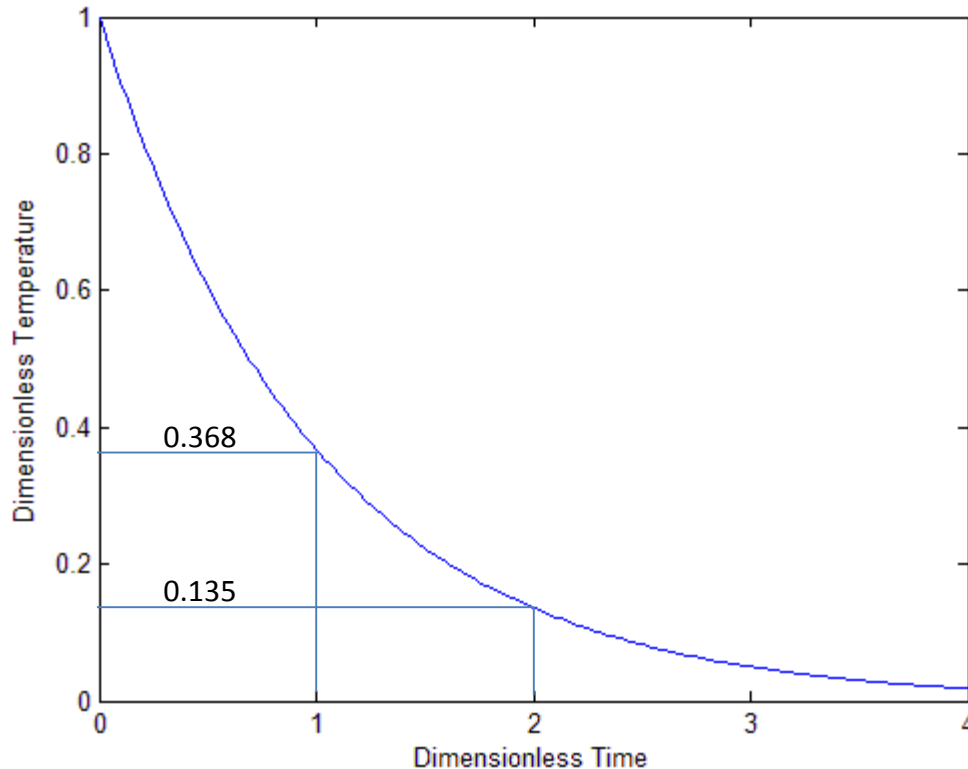
The left side of the solution, $(T - T_{\infty})/(T_0 - T_{\infty})$, also is dimensionless. We can label it as a dimensionless temperature θ , defined as

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty}$$

Thus, the solution can be written in simple form as follows.

$$\theta(\tau) = e^{-\tau}$$

A sketch of this solution is shown below.



The first observation from the sketch is that the dimensionless temperature decreases exponentially with time toward a value of zero; of course, this value of zero is achieved only at infinite time. When the dimensionless temperature is zero, that means that the mass has achieved the same temperature as the surrounding fluid. In practice, we do not have infinite time, and one might wonder how long it will take for the temperature of the mass to become sufficiently close to T_∞ that it can be regarded as T_∞ for practical purposes. It is seen from the sketch that when the dimensionless time has a value of unity, the dimensionless temperature is approximately 0.368. Thus the difference between T and T_∞ is about 36.8% of the difference between the initial temperature T_0 and T_∞ . By the time we reach a dimensionless time value of 2, this ratio has decreased to approximately 13.5%. If you are wondering how long it takes for the ratio to be 1%, the answer is $\tau \approx 4.61$. Thus, by the time the dimensionless time is approximately 5, we can assume that the temperature of the mass is practically indistinguishable from the temperature of the surrounding fluid.

Validity of the Assumption of Lumped Thermal Capacity

Our initial assumption was that the temperature variations within the mass should be small, when compared with the temperature difference from the surroundings. We said that in that case, we can neglect such small variations and assume the entire mass to be at a single temperature. From our earlier results on conduction, we know that for temperature variations to be small, the resistance to conduction must be small. Thus, it can be seen that the assumption we made is good when

$$\frac{\text{Internal resistance}}{\text{External resistance}} \ll 1$$

where the symbol \ll is used to indicate the phrase: “is much less than”

Suppose the mass is a solid sphere of radius R and thermal conductivity k_s . Then, the length of the path for heat conduction within the sphere is R and the internal resistance can be approximately represented by $R/(k_s A)$. The external resistance to heat transfer is the convective resistance $1/(hA)$. Therefore, our requirement for the model to be valid can be cast approximately as

$$\frac{R/(k_s A)}{1/(hA)} \ll 1 \quad \text{or} \quad \frac{hR}{k_s} \ll 1$$

You can verify that the group $(hR)/k_s$ is dimensionless. It is called the Biot Number and abbreviated as Bi .

$$Bi = \frac{hR}{k_s}$$

for a sphere. If the solid has some other shape, a suitable characteristic length that represents the conduction path can be used instead of the radius of the sphere used here. We have specifically used the subscript s for the thermal conductivity to specify that it is the thermal conductivity of the solid and not that of the fluid.

Thus, the condition for the lumped thermal capacity model to be a good representation of the actual heat transfer system is $Bi \ll 1$.