# Thermal Analysis of a Steady State Heat Exchanger

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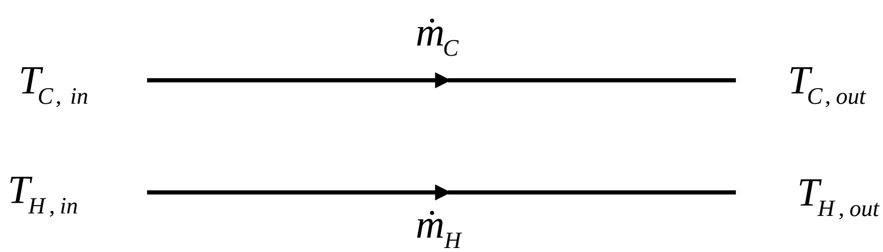
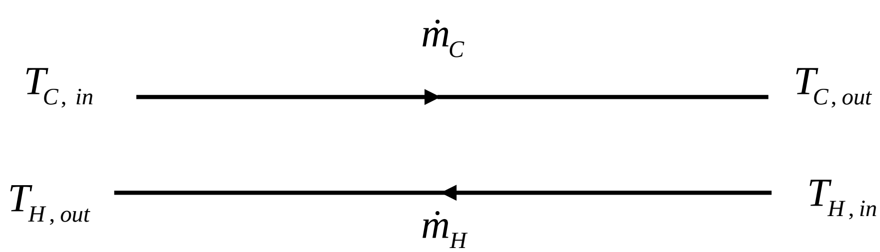
As discussed in the textbook, heat exchangers are used widely in the chemical process industries. Typically, two streams flow through the heat exchanger, and heat is transferred from the hot stream to the cold stream. The hot stream enters the heat exchanger at a relatively high temperature, and leaves it at a lower temperature, transferring heat to the cold stream, which enters at a relatively low temperature, and leaves the exchanger at a higher temperature. In the analysis presented here, we consider steady state operation, and we assume that the heat lost to the surroundings is negligible compared with the rate at which heat is exchanged between the two streams, labeled the heat load $Q$. While there are numerous ways of arranging the flow of each stream through the heat exchanger, for modeling purposes we simplify these to two possible patterns. One is termed a cocurrent heat exchanger, while the other is termed a countercurrent heat exchanger. In the cocurrent mode, both the hot and cold streams enter the heat exchanger at one end, and leave at the opposite end. In the countercurrent mode, the streams enter at opposite ends of the heat exchanger. The two modes are illustrated below.

Cocurrent Mode

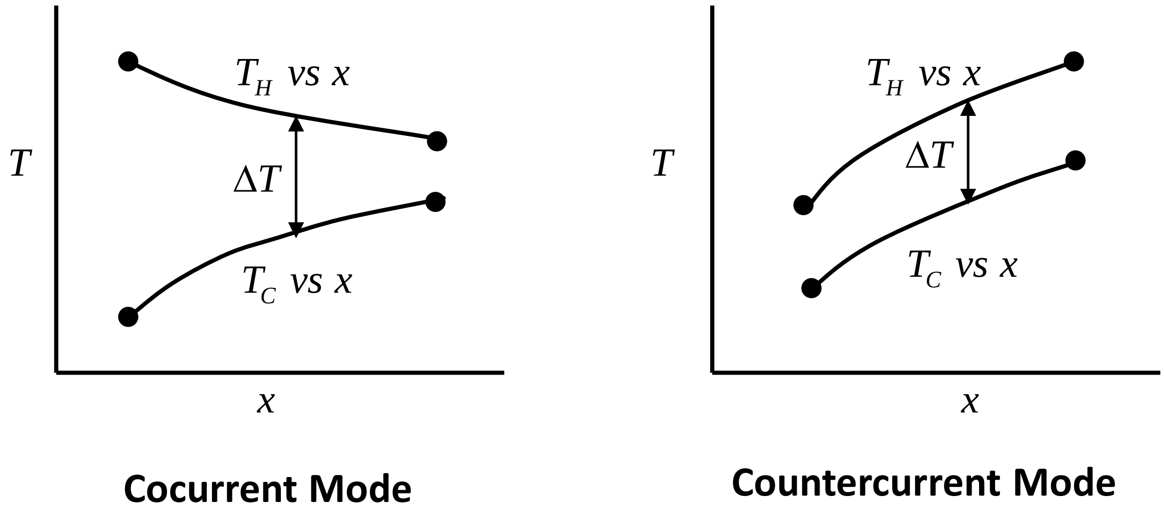



Countercurrent Mode




In the sketches, the symbols $\dot{m}_C$ and $\dot{m}_H$ represent the mass flow rates of the cold and hot streams, respectively. The symbols used for the incoming and outgoing temperatures of the two streams are self-explanatory, with the subscript $C$ representing the cold stream and the subscript $H$ representing the hot stream.

Qualitative sketches of the temperature distributions in each mode are provided below.




In both cases, note that the hot fluid at any axial location in the heat exchanger is at a higher temperature than the cold fluid. In the cocurrent mode, the temperature difference $\Delta T$ is largest where the two streams enter the heat exchanger, and decreases to its smallest value at the other end. The actual rate of change in $\Delta T$ will be governed by local heat exchange rates. In the countercurrent mode, the temperature difference does not vary as much along the heat exchanger as in the cocurrent mode. In the countercurrent mode, $\Delta T$ can be larger at either end, whereas in a cocurrent heat exchanger it is always largest at the entry point of the hot and cold streams.

**Steady State Energy Balance**

For both modes of operation, we can write a steady state energy balance as follows.

$$Q = \dot{m}_C \left( \hat{H}_{C,out} - \hat{H}_{C,in} \right) \; = \; \dot{m}_H \left( \hat{H}_{H,in} - \hat{H}_{H,out} \right)$$

Here, $\hat{H}$ stands for the enthalpy per unit mass of each stream, and the subscripts $H$ and $C$ represent the hot and cold streams, respectively. As noted earlier, we have assumed that all the energy leaving the hot stream per unit time enters the cold stream, and neglect heat loss to the surroundings.

Because the enthalpy is a function of pressure $p$ and temperature $T$, we can write the following result for a differential change in enthalpy.

$$d\hat{H} = \left( \frac{\partial \hat{H}}{\partial T} \right)_p dT + \left( \frac{\partial \hat{H}}{\partial p} \right)_T dp$$

The rate of change of enthalpy with pressure at constant temperature can be assumed to be negligible, so that we can approximate the above relationship by

$$d\hat{H} = \left( \frac{\partial \hat{H}}{\partial T} \right)_p dT = c_p \; dT$$

where $c_p$ is the specific heat of the fluid in the stream at constant pressure. Neglecting small variations in the specific heat between the incoming and outgoing streams (using an average value if needed), we can rewrite the steady state energy balance as follows.

$$Q = \dot{m}_C \; c_{p,C} \left( T_{C,out} - T_{C,in} \right) \; = \; \dot{m}_H \; c_{p,H} \left( T_{H,in} - T_{H,out} \right)$$

We now define flow thermal capacities $C_C = \dot{m}_C c_{p,C}$ for the cold stream and $C_H = \dot{m}_H c_{p,H}$ for the hot stream and cast the steady state energy balance as

$$\boxed{Q = C_C \left( T_{C,out} - T_{C,in} \right) \; = \; C_H \left( T_{H,in} - T_{H,out} \right)}$$

**Steady State Rate Equation**

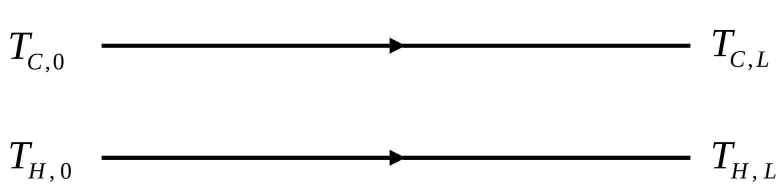
Now, we proceed to develop a rate equation for a heat exchanger. Note that the temperature difference $\Delta T$, which is the driving force for heat transfer, varies along the length of the heat exchanger. We would like to be able to write

$$Q = UA \, \Delta T_m$$

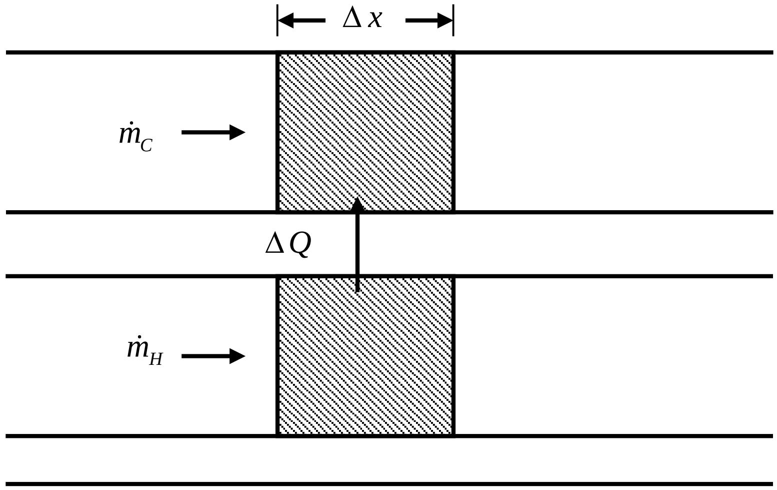
where $U$ is an average overall heat transfer coefficient, $A$ is the area of the heat transfer surface, and $\Delta T_m$ is some type of average value of the temperature difference $\Delta T$. But what is this average? To answer this question, we must perform a detailed analysis. We do this for the cocurrent mode below.

**Rate Equation for a Cocurrent Heat Exchanger**

We begin with a sketch showing the temperatures of the two streams at the ends of a cocurrent heat exchanger. For convenience in the analysis, the left end is identified with the subscript $0$, and the right end with the length of the heat exchanger, $L$. The coordinate along the length of the heat exchanger is $x$, which varies from $0$ to $L$.



If we take an elemental section of the heat exchanger that is sketched below,




we can write

$$\Delta Q = C_C \left[ T_C \left( x + \Delta x \right) - T_C \left( x \right) \right] = - C_H \left[ T_H \left( x + \Delta x \right) - T_H \left( x \right) \right]$$

Note the difference in signs for the hot and cold fluids. By definition, the change $\Delta Q$ is assumed positive. The cold fluid and hot fluid both enter at the left end. As the cold fluid proceeds along the length of the heat exchanger, it gains heat so that its temperature increases with increasing $x$. Therefore, $T_C\left(x+\Delta x\right)$ is larger than $T_C\left(x\right)$. In contrast, the hot fluid is losing heat as it flows along the length of the heat exchanger, so that its temperature decreases with increasing $x$. This means that $T_H\left(x+\Delta x\right)$ is smaller than $T_H\left(x\right)$, and we must introduce a minus sign in front of the expression to obtain a positive value for $\Delta Q$ . You can understand this also by examining the qualitative temperature profiles that we sketched earlier. In contrast to a cocurrent heat exchanger, for one operating in the countercurrent mode, both the hot and cold fluid temperatures increase with increasing $x$, and we would use a positive sign for both the hot and cold fluid energy changes in the result for $\Delta Q$ .

The rate equation at this location in the heat exchanger is

$$\Delta Q = UP\Delta x\left(T_H - T_C\right)$$

where $P$ is the perimeter of the heat transfer surface, so that the heat transfer area in this elemental section is $\Delta A = P\,\Delta x$. In the analysis, we shall assume $P$ to be constant along the length of the heat exchanger. Equating the results for $\Delta Q$ from the energy balance for the cold stream and the rate equation, we obtain

$$UP\Delta x\left(T_H - T_C\right) = C_C \left[ T_C \left( x + \Delta x \right) - T_C \left( x \right) \right]$$

Divide throughout by $C_C\ \Delta x$, and take the limit as $\Delta x \to 0$.

$$\underset{\Delta x \to 0}{Limit}\left[\frac{T_C\left(x+\Delta x\right)-T_C\left(x\right)}{\Delta x}\right] = \frac{UP}{C_C}\left(T_H - T_C\right)$$

which becomes the following differential equation.

$$\frac{dT_C}{dx} = \frac{UP}{C_C}\left(T_H - T_C\right) = \frac{UP}{C_C}\,\Delta T \qquad (1)$$

where the temperature driving force $\Delta T = T_H - T_C$ at any given location in the heat exchanger.

In a like manner, we can write

$$UP\Delta x\left(T_H - T_C\right) = -\,C_H\left[T_H\left(x+\Delta x\right) - T_H\left(x\right)\right]$$

and go through the same process to obtain the differential equation

$$\frac{dT_H}{dx} = -\frac{UP}{C_H}\left(T_H - T_C\right) = -\frac{UP}{C_H}\,\Delta T \qquad (2)$$

Now, subtract Equation (1) from Equation (2). Recognizing that $d\left(T_H - T_C\right)/dx = d\left(\Delta T\right)/dx$ , we obtain

$$\boxed{\frac{d\left(\Delta T\right)}{dx} = -UP\left(\frac{1}{C_C} + \frac{1}{C_H}\right)\Delta T}$$

This is a differential equation for the rate of change of the temperature driving force along the length of the heat exchanger. Treating the overall heat transfer coefficient $U$ as a constant along the length of the heat exchanger, we can integrate this equation between the two ends of the heat exchanger as follows.

$$\int_{\Delta T_0}^{\Delta T_L} \frac{d\left(\Delta T\right)}{\Delta T} = -UP\left(\frac{1}{C_C} + \frac{1}{C_H}\right)\int_0^L dx = -UPL\left(\frac{1}{C_C} + \frac{1}{C_H}\right) = -UA\left(\frac{1}{C_C} + \frac{1}{C_H}\right)$$

where the product $PL$ is the total heat transfer area $A$, and $\Delta T_0$ and $\Delta T_L$ are the values of the driving force $\Delta T$ at the two ends of the heat exchanger. The integral on the left side of the above equation is seen to be $\ln\left(\Delta T_L / \Delta T_0\right)$ so that our result can be recast as follows.

$$\ln\left(\frac{\Delta T_L}{\Delta T_0}\right) = -UA\left(\frac{1}{C_C}+\frac{1}{C_H}\right) \qquad (3)$$

From the steady state energy balance, we can also write the heat load as

$$Q = C_C\left(T_{C,L} - T_{C,0}\right) \;=\; C_H\left(T_{H,0} - T_{H,L}\right)$$

Therefore,

$$\frac{1}{C_C}+\frac{1}{C_H} \;=\; \frac{1}{Q}\left[\left(T_{C,L}-T_{C,0}\right)+\left(T_{H,0}-T_{H,L}\right)\right] \;=-\; \frac{1}{Q}\left[\left(T_{H,L}-T_{C,L}\right)-\left(T_{H,0}-T_{C,0}\right)\right]$$

$$= -\; \frac{1}{Q}\left[\Delta T_L \;-\Delta T_0\;\right]$$

Let us substitute this result for the sum $\frac{1}{C_C}+\frac{1}{C_H}$ in Equation (3).

$$\ln\left(\frac{\Delta T_L}{\Delta T_0}\right) = -UA\left(\frac{1}{C_C}+\frac{1}{C_H}\right) = \;\frac{UA}{Q}\left[\Delta T_L \;-\Delta T_0\;\right]$$

Rewrite this result as

$$Q \;=\; UA\;\frac{\left(\Delta T_L \;-\Delta T_0\right)}{\ln\left(\dfrac{\Delta T_L}{\Delta T_0}\right)}$$

or

$\boxed{Q = UA\,\Delta T_{lm}}$ where $\Delta T_{lm} = \left(\Delta T_L \;-\Delta T_0\right)/\ln\left(\Delta T_L/\Delta T_0\right)$. Thus, we find that the heat load of the heat exchanger can be obtained from a simple rate equation that uses a constant overall heat transfer coefficient and the area of the heat exchanger by employing an average temperature driving force that happens to be the log-mean of the values of the driving force at the two ends of the heat exchanger.

It is reasonable to assume that the overall heat transfer coefficient has a single average value throughout the heat exchanger, as we did in this analysis. This is adequate in most situations. If the overall heat transfer coefficient varies significantly along the length of the heat exchanger, its variation can be accommodated in the analysis.

From the homework assignment, you'll learn that precisely the same rate equation is obtained in the countercurrent mode of operation. By spreading out the driving force more evenly across the heat exchanger, the countercurrent mode results in a larger log-mean average driving force, all else being the same. Therefore, the countercurrent mode of operation will require a smaller heat transfer surface area, and therefore, it is almost always the preferred mode.