The Loneliest Number

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We examined the "Friendly Integer" problem first stated in [3]. A pair (m, n) is called a friendly pair if $\sigma(n)/n = \sigma(m)/m$ where $\sigma(n)$ is the sum of all the divisors of n. In this case, it is also common to say that m is a friend of n, or simply that m and n are friends. For convience, we define the function $\hat{\sigma}(n) = \sigma(n)/n$. This is often called the abundacy or the index of a number. Perfect numbers have abundancy 2, and thus are all friends. Numbers with abundancy less than 2 are often called deficient, while numbers whose abundancy are greater than 2 are called abundant [6]. The original problem was to show that the density of friendly integers in $\mathbb{N}$ is unity, and the density of solitary numbers (numbers with no friends) is zero. Also note, that any variables will be positive integers unless stated otherwise. The two letters p and q will represent prime integers throughout the paper.

We used two main approaches: one was an analysis of the $\hat{\sigma}(n)$ function, while the other used number theoretic arguments to find a representation for a friend of 10. In [3], it was shown that 10 is the smallest number where it is unknown whether there are any friends of it. We will assume a basic understanding of the function $\sigma(n)$. This can be found in [2]. For more on number theoretic techniques, see [1].

We discovered many useful properties of $\hat{\sigma}(n)$:

1. $\hat{\sigma}(nm) = \hat{\sigma}(n)\hat{\sigma}(m)$ if $gcd(m, n) = 1$
2. $\hat{\sigma}(n) > 1$ for $n > 1$
3. For prime p, integers $a > b$, $\hat{\sigma}(p^a) > \hat{\sigma}(p^b)$
4. For primes $p < q$, $\hat{\sigma}(p^a) > \hat{\sigma}(q^a)$

Proof.

1. $\sigma(n)$ is weakly multiplicative, therefore $\sigma(nm)/nm = (\sigma(n)/n)(\sigma(m)/m) = \hat{\sigma}(n)\hat{\sigma}(m)$ when $gcd(m,n) = 1$
2. This follows directly from $\sigma(n) > n$ for $n > 1$
3. In [3], it was shown that $\hat{\sigma}(p^{a+1}) > \hat{\sigma}(p^a)$, this is a natural generalization
4. Show that $\hat{\sigma}(p^a)$ decreases for larger primes, suppose $q > p$:

$$\hat{\sigma}(p^a) - \hat{\sigma}(q^a) = \frac{1+p+p^2+\ldots+p^a}{p^a} - \frac{1+q+q^2+\ldots+q^a}{q^a}$$

It is enough to show that

$$q^a(1+p+p^2+\ldots+p^a) - p^a(1+q+q^2+\ldots+q^a) < 0$$

Regroup the terms

$$(q^a - p^a) + (q^a p - p^a q) + (q^a p^2 - p^a q^2) + \ldots + (q^a p^a - p^a q^a)$$
$$= \underbrace{(q^a - p^a)}_{<0} + qp\underbrace{(q^{a-1} - p^{a-1})}_{<0} + q^2p^2\underbrace{(q^{a-2} - p^{a-2})}_{<0} + \ldots + q^a p^a \underbrace{(1-1)}_{=0} < 0$$

□

Some of these properties can be generalized further, but these will be our building blocks to prove bigger results.

Proposition 1. *$\hat{\sigma}(n) < \hat{\sigma}(an)$ for $a > 1$*

Proof. In general, a can share prime factors with n. Let $a = lm$ where $gcd(a,n) = l, gcd(m,n) = 1$. We thus have $\hat{\sigma}(an) = \hat{\sigma}(ln)\hat{\sigma}(m)$ by property 1. Property 2 gives

$$\hat{\sigma}(an) = \hat{\sigma}(ln)\hat{\sigma}(m) > \hat{\sigma}(ln)$$

Now property 3 gives us

$$\hat{\sigma}(ln) > \hat{\sigma}(n)$$

Thus

$$\hat{\sigma}(an) > \hat{\sigma}(n)$$

□

Corollary 1. *When* $a = n^b$ *we get a generalization of property 3:* $\hat{\sigma}(n^j) < \hat{\sigma}(n^k)$ *for* $j < k$*.*

Note that property 4 is not true for general $x > y$.

Using these tools, we analyzed whether 10 has a friend and what forms that friend can and cannot take. Notice that $\hat{\sigma}(10) = 9/5$. We can easily deduce that a friend of 10 must be of the form $n = 5^a m$ otherwise $\hat{\sigma}(n)$ could not reduce to the fraction 9/5.

Lemma 1. *A friend of* m *cannot be a multple of* m*. That is* $\hat{\sigma}(m) \neq \hat{\sigma}(am)$ *for* $a > 1$*.*

Proof. This follows directly from Propostion 1. Since am is a multiple of m, $\hat{\sigma}(am) > \hat{\sigma}(m)$, so $\hat{\sigma}(am) \neq \hat{\sigma}(m)$. ☐

Corollary 2 (Even Corollary)**.** *A friend of* 10 *cannot be of the form* $n = 2^a 5^b m$*. Thus a friend of 10 cannot be an even integer.*

Proof. For $a, b > 1$, $2^a 5^b$ is a multiple of 10, and thus so is $n = 2^a 5^b m$. Therefore n is not a friend of 10. A friend of 10 must be of the form $5^b m$, so a friend of 10 cannot be an even integer. ☐

Corollary 3. *A friend of* 10 *must be the square of some number:* $n = 5^{2b} m^2$*.*

Proof. Suppose $\hat{\sigma}(n) = 9/5$ and $n = 5^b m$, $m = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ then

$$\frac{\sigma(n)}{n} = \frac{9}{5}$$

$$5\sigma(5^b)\sigma(p_1^{e_1})\sigma(p_2^{e_2}) \dots \sigma(p_k^{e_k}) = (9)5^b p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$$

$$5(1 + 5 + \dots + 5^b)(1 + p_1 + \dots + p_1^{e_1}) \dots (1 + p_k + \dots + p_k^{e_k}) = (9)5^b p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$$

From the Even Corollary, we have $p_i > 2$ for any $i \leq k$. So the right side must be odd. To obtain this, we must have that every sum on the left side is also odd. Since each p_i^x is odd, we must have an odd number of terms in the sum for the whole sum to be odd. To obtain this, each e_i and b must be even. Thus n must be the square of some number. ☐

Proposition 2. *If* $\hat{\sigma}(n) = 9/5$*, then* $3 \nmid n$*.*

Proof. From the even corollary, we have that if $\hat{\sigma}(n) = 9/5$ and $3 \mid n$, then $n = 3^{2a}5^{2b}m^2, 2, 3, 5 \nmid m$. It is easy to verify that $\hat{\sigma}(3^4 5^2)$ and $\hat{\sigma}(3^2 5^4) > 9/5$. Combining this with Lemma 1, we have that $\hat{\sigma}(3^4 5^2 m^2)$, $\hat{\sigma}(3^2 5^4 m^2) > 9/5$. Thus, the problem reduces to a single case: $\hat{\sigma}(3^2 5^2 m^2) = 9/5$.

$$\begin{aligned} 5\sigma(3^2)\sigma(5^2)\sigma(m^2) &= 9(3^2)(5^2)m^2 \\ 13(31)\sigma(m^2) &= (3^4)(5)m^2 \end{aligned}$$

We can see that $13, 31 \mid m^2$, thus $m^2 = 13^c 31^d k^2$ where $2, 3, 5, 13, 31 \nmid k$. We will divide by 13,31 immediately.

$$\begin{aligned} \sigma(13^c)\sigma(31^d)\sigma(k^2) &= 3^4(5)(13^{c-1})(31^{d-1})k^2 \\ \frac{\sigma(k^2)}{k^2} &= \frac{405}{\underbrace{\frac{\sigma(13^c)}{13^{c-1}}}_{\geq 14}\underbrace{\frac{\sigma(31^d)}{31^{d-1}}}_{\geq 32}} < \frac{405}{448} < 1 \end{aligned}$$

This is a contradiction with property 2 for $\hat{\sigma}(k^2)$. Hence, $3 \nmid n$ whenever n is a friend of 10. □

Proposition 3. $\lim_{k\to\infty} \hat{\sigma}(p^k) = p/(p-1)$

Proof.

$$\begin{aligned} \lim_{k\to\infty} \frac{\sigma(p^k)}{p^k} &= \lim_{k\to\infty} \frac{p^{k+1}-1}{p^k(p-1)} \\ &= \lim_{k\to\infty} \frac{p - 1/p^k}{p-1} = \frac{p}{p-1} \end{aligned}$$

□

Proposition 4. *If $\hat{\sigma}(n) = 9/5$, then $n = 5^{2a}p_1^{e_1}\dots p_m^{e_m}$ where $m \geq 4$.*

Proof. Using the previous proposition and properties 3 and 4, we will construct the largest value of $\hat{\sigma}(n^k)$ with 4 distinct primes. Let $n = (5)(7)(11)(13)$. Here is a largest value with four distinct primes because from proposition 2, we have that $p_i \geq 7$. To maximize the value of $\hat{\sigma}(n^k)$ we let $k \to \infty$. Hence,

$$\lim_{k\to\infty} \hat{\sigma}(n^k) = (5/4)(7/6)(11/10)(13/12) = \frac{1001}{576} < \frac{9}{5}$$

So there must be at least 5 distinct primes in the factorization of n. □

The rest of the paper will be refinements in our representation of n, where n is a friend of 10.

Proposition 5. *If $\hat{\sigma}(n) = 9/5$, then $n = 5^{2a}p_1^{e_1}\dots p_k^{e_k}$ where $k \geq 5$.*

Proof. We can use the same technique as in the last proposition to show that only 3 cases could work if n were represented as 5 distinct primes. These are $n = (5^a 7^b 11^c 13^d 17^f)^2$, but this does not work because the smallest it could be is: $\hat{\sigma}(5^2 7^2 11^2 13^2 17^2) > 9/5$. Case 2 gives $n = (5^a 7^b 11^c 13^d 19^f)^2$, but this does not work becuase the smallest it could be is: $\hat{\sigma}(5^2 7^2 11^2 13^2 19^2) > 9/5$. The final case takes a little more work: $n = (5^a 7^b 11^c 13^d 23^f)^2$. We can see that if $a > 1$, then $\hat{\sigma}(n) > 9/5$, so $a = 1$. Let us examine when $\hat{\sigma}(n) = 9/5$, then

$$5\sigma(5^2)\sigma(7^{2b})\sigma(11^{2c})\sigma(13^{2d})\sigma(23^{2f}) = 9(5^2)(7^b 11^c 13^d 23^f)^2$$
$$31\sigma(5^2)\sigma(7^{2b})\sigma(11^{2c})\sigma(13^{2d})\sigma(23^{2f}) = 9(5^2)(7^b 11^c 13^d 23^f)^2 = l$$

Clearly, $31 \nmid l$. Since the left hand side is some integer, this results in a contradiction. Hence a friend of 10 must be composed of at least 6 distinct primes. □

Proposition 6. *If $\hat{\sigma}(n) = 9/5$, then $n = 5^{2a}p_1^{e_1}\dots p_i^{6e_i+2}\dots p_k^{e_k}$ where $k \geq 5$ and $p_i \equiv 1 \mod 3$ for some $i, 1 \leq i \leq k$.*

Proof. Suppose we had a friend $\hat{\sigma}(n) = 9/5$, then the resulting equation occurs for $n = 5^{2a}m^2$

$$5\sigma(5^{2a})\sigma(m^2) = 9(5^{2a})m^2$$
$$2\sigma(5^{2a})\sigma(m^2) \equiv 0 \mod 3$$

Let $q_1 \equiv 1 \mod 3$, then $\sigma(q_1^{6x+2}) \equiv 0 \mod 3, \sigma(q_1^{6x+4}) \equiv 2 \mod 3, \sigma(q_1^{6x}) \equiv 1 \mod 3$. Let $q_2 \equiv 2 \mod 3$, then $\sigma(q_2^{2y}) \equiv 1 \mod 3$. Clearly for the above stated equation to be true, we must have some $p_i \equiv 1 \mod 3$ in the factorization of m. Moreover, it must be of the form p_i^{6x+2}. □

Proposition 7. *If $\hat{\sigma}(n) = 9/5$, then $n = 5^{2a}p_1^{e_1}\dots p_i^{6e_i+2}\dots p_j^{2(2e_j+1)}\dots p_k^{e_k}$ where $k \geq 5$ and $p_i \equiv 1 \mod 3$ for some $i, 1 \leq i \leq k$ and either $a = 2x$ or $\exists p_j, 1 \leq j \leq k$ such that $p_j \equiv 1 \mod 4$.*

Proof. The arguments are identical to those of the previous proposition, except mod 4. Notice that an either/or condition results instead of a single fact. □

Just for fun, we introduce a new definition.

Definition 1 (Theoretical Friend)**.** A sequence n_k is a **theoretical friend** of m if: $\lim_{k\to\infty} \hat{\sigma}(n_k) = \hat{\sigma}(m)$.

Proposition 8. *10 has at least one theoretical friend, namely* $n_k = 3^k 5$.

Proof.

$$\lim_{k\to\infty} \hat{\sigma}(n) = \lim_{k\to\infty} \frac{\sigma(3^k)}{3^k}\frac{\sigma(5)}{5}$$
$$= \left(\frac{3}{2}\right)\left(\frac{6}{5}\right) = \frac{9}{5} = \hat{\sigma}(10)$$

□

For further reading on the topic of $\hat{\sigma}(n)$ and $\sigma(n)$, see [5] and [4]. See [5] for information concerning when $\sigma(n) = k$ has exactly m solutions (Sierpiński Conjecture). See [4] for a more indepth study of $\hat{\sigma}(n)$ and on the distribution and density of numbers of this form.

Literatur

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