

CS447 Computer Algorithms. Notes 4: More Asymptotics. Fall 2021

We will use the asymptotic notations (Big-Oh, Big-Ω, Big-Θ, little-oh, little-ω) to *order* functions. Consider the following cast of functions on n (which is often the input size to algorithms):

$$1, \quad \log n, \quad n, \quad n \log n, \quad n^2, \quad 2^n, \quad n!$$

Recall the following identities for arithmetic and geometric summations:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}, \quad \sum_{k=0}^n a^k = \frac{a^{n+1} - 1}{a - 1}, \quad \text{for positive } a \neq 1$$

Basic observations

1. All logarithms are created equal: $\log_b n = \Theta(\log_2 n)$.

This holds since by the law of logarithms, we have

$$\log_b n = \frac{\log_2 n}{\log_2 b}$$

and note that $\log_2 b$ is a constant.

2. Swapping around the logarithm: $a^{\log_b c} = c^{\log_b a}$.

This holds since by taking logarithms of both sides, we get

$$(\log_b c)(\log_b a) = (\log_b a)(\log_b c).$$

3. Logarithm is slower than linear: $\log n = o(n)$.

True since

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

by applying l'Hopital's rule.

(T/F) Polynomial is slower than exponential: $n = o(2^n)$.

(T/F) Poly-logarithmic is slower than sub-linear: $(\log n)^M = o(n^{\hbar})$, where M is the number of particles in the universe and $\hbar$ is the Planck constant.

4. The logarithm of factorial equals optimal sorting bound: $\log(n!) = \Theta(n \log n)$.

First, we observe that

$$(n/2)^{n/2} \leq n! \leq n^n.$$

Taking logarithm (which is a monotonically increasing function), we get

$$\frac{n}{2} \log \frac{n}{2} \leq \log(n!) \leq n \log n.$$

It can be shown that $(n/2) \log(n/2) = \Omega(n \log n)$ (Explain why). This combined with the second inequality (which trivially shows $\log(n!) = O(n \log n)$) yields $\log(n!) = \Theta(n \log n)$.

5. little-oh is transitive: $f(n) = o(g(n))$ and $g(n) = o(h(n))$ implies $f(n) = o(h(n))$.
6. Polynomials are ordered by power: $n^a = O(n^b)$ if and only if $a \leq b$.
7. Exponentials are ordered by base: $a^n = O(b^n)$ if and only if $0 < a \leq b$.

8. Arithmetic sum of linearly growing terms is quadratic: $\sum_{k=1}^n k = \Theta(n^2)$.

We bound the sum from below (omit first half and replace second half by middle) and above (replace all by maximum) to get

$$\frac{n}{2} \times \frac{n}{2} \leq \sum_{k=1}^n k \leq n \times n.$$

9. Geometric sum of linearly growing terms is exponential: $\sum_{k=0}^n 2^k = \Theta(2^n)$.

We bound the sum from below (use maximum element) and above (the maximum element dominates the sum of rest) to get

$$2^n \leq \sum_{k=0}^n 2^k \leq 2^n + \sum_{k=0}^{n-1} 2^k < 2^{n+1}.$$

10. Harmonic sum: an arithmetic sum of reciprocals of linearly growing terms: $\sum_{k=1}^n \frac{1}{k} = \Theta(\log n)$.

We bound the sum from below and above using integrals:

$$\ln(n+1) = \int_1^{n+1} \frac{dx}{x} \leq \sum_{k=1}^n \frac{1}{k} \leq 1 + \int_1^n \frac{dx}{x} = 1 + \ln n$$

Show: $\ln(n+1) = \Omega(\log n)$ and $1 + \ln n = O(\log n)$.